Electro-Magnetism Is

A Fundamentally Fractal Fibonacci Fourier Field

by

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Abstract. We offer a purely computational and combinatorial explanation for Coldea *et al.*'s 2010 report of having measured the golden mean in a quantum system. Our method employs the roots of projectors (in the discrete and finite geometric algebra $\mathscr{G}_{5,0}$) to capture both the dimensionality (*3+1*d) and the detailed structure of the electro-magnetic field, including Majorana fermions (with fresh details). The pattern of growth as the field expands from its source displays the Fibonacci sequence $F = 1, 1, 2, 3, 5, \ldots$, where $\lim_{n\to\infty} \frac{F_{i+1}}{F_i} = \varphi$, the golden mean. The Fibonacci sequence and the golden mean are thus guiding principles at and from the very root of our universe.

Keywords: Fibonacci, golden mean, distributed, self-organizing, topological, computational, combinatorial, hierarchical, fractal, systems, Fourier, Parseval, coordinate free, geometric algebra, tauquinion, tauquernion, enquernion, entangled quaternion, Majorana.

0. Introduction - Quantum Systems Are Distributed Systems

A long-standing interest in computer science is how to define and construct *distributed* systems - systems consisting of many more-or-less independent asynchronous computational processes distributed over a number of hosts, that carry out coherent system-wide computations with little or no central coordination. Due to their complexity, there is furthermore much interest in making distributed systems that are *self-organizing*. After all, Nature is unboundedly complex, and we can't write code for every eventuality.

Atoms, crystals, beehives & anthills, ecologies, weather systems, indeed all the works of Nature are such distributed systems, but despite their ubiquity and familiarity, the fundamental principles governing their design and operation are subtle and elusive. It's as if the whole universe were running on an invisible global operating system, one whose goals are namely (like any good operating system) to get everything done, and to be invisible while doing so.

In particular, quantum systems are distributed systems, and in this paper we apply our self-organizing distributed system analysis tools to Coldea *et al*'s finding [1,10], and expose its underlying computational mechanism.

Our algebraic representation of computation - using the real geometric (Clifford) algebras $\mathscr{G}_{n,0} = \mathscr{G}_n$ - shows that the defining property of distributed systems is that they are $wave-like$, in that, conceptually, a wave is everywhere $\ldots$ and yet, simultaneously, not in any one place in particular. Being wave-like means that distributed systems are *space-like*, viz. rotation around a circle, the *plane* and not the *line* (= individual sequential processes).

Sequential processes, such as those generated by typical programs, are, in contrast, *time-like*. These (so-to-speak) wend their way through the above "distributed space", *a la* relativity theory's reference frames. We have little more to say about sequential processes here, but see [6].

The various special properties of our algebraic representation of distributed computation - its discreteness yields combinatorics and structure, its graded hierarchy collapses structural complexity, the tauquinions (see below) specify field structure, and more - allow us to address the phenomenon observed by Coldea *et al.* in a novel manner.

The computational interpretation that we place on our algebra is simple: *concurrent processes - being independent - are considered to be orthogonal to each other*. Two concurrent 1-bit-of-state processes a, b are written as $a + b$, and it turns out (unobviously from the present explanation) that the ensuing multiplicative anti-commutativity, $ab = -ba$, exactly tracks everything that happens (bit by bit, as 'twere) when two such processes are co-present. We thus build more complex processes (like ab) from these 1-bit primitive processes; that is, every expression in the algebra is either a process or built from same (and some expressions are more interesting than others).

We draw the algebra's scalar coefficients from $\mathbb{Z}_3 = \{0, 1, 2\} \mapsto \{0, 1, -1\}$ for nine reasons (so far):

(1) The binary feel of ± 1 is useful, eg. it makes ab into an *xor* (at the scalar level), and dovetails nicely with information theory's requirements;
(2) $\mathbb{Z}_3 = \{0, 1, -1\}$ means *no counting*, thus subverting sequentiality at its very root. Counting is replaced by simple *distinction*: same or different. Structure comes from the distinction *co-occur/exclude* on process states [7, §7.2];
(3) Generality requires the simplest possible atoms - 1-bit processes;

(4) Zero no longer wears two hats - *Void* and "the opposite of 1" - as it does in the usual $\mathbb{Z}_2 = \{0,1\}$ binary system;
(5) The introduction of a minus *number* into the basic algebra (versus $\mathbb{Z}_2$) [1] makes the transition to geometric algebra's vector world easy;
(6) Geometric algebra's various physics-relevant isomorphs (eg. the quaternion, Pauli, Grassmann, and Dirac algebras), when expressed in the extremely minimal $\mathbb{Z}_3$ algebra, yield a *very* tight fit, eliminating most questions of correct interpretation;
(7) This same minimality makes exhaustive searches of $\mathscr{G}_3$ and $\mathscr{G}_4$ possible, and such searches are our main source of data, eg. for entropy calculations [7];
(8) The extreme minimalism imposed by $\mathbb{Z}_3$ promotes the exhibition of much symmetry that is implicit and hidden in the turmoil of *multiplicities* of identicals that one finds in larger number systems; [2]
(9) Since physical three-ness is both common and deep (three spatial dimensions, three particle generations, three quark pairs, etc.), the match with $\mathbb{Z}_3$ further focuses the algebra's precision of expression.

Further discussion of the algebra appears below.

Finally, because our analysis is purely combinatorial, over a universe of arbitrary processes, it is independent of any particular physical theory.

0.1 *Earlier Work*

This paper is logically an addendum to [7], in that it follows directly from, and fits directly into, this prior work. In that work, we identified a novel set of isomorphs $\tau = \{ab-cd, ac+bd, ad-bc\}$, dubbed *TauQuernions*, to the classical *quaternions* $Q = \{ab, bc, ca\}$ - the very definition of 3d space. The tauquernions are novel not only because they are new on the scene, but rather, especially, because they are $time-like$.

That is, tauquernions describe 3d space in *exactly* the same way quaternions do, *but* while $(ab)^4 = +1$, one gets $(ab-cd)^4 = -1-abcd$ for the corresponding tauquernion. [3] $(-1-abcd)^2 = -1-abcd$ is idempotent, just like +1, so we're in a dual algebra ... and *in* that dual place, "$+1$"$= -1-abcd$ is *also* a projector, a measurement operator. That is, it is $time-like$, so when 3-space is constructed using tauquernions, the quaternions' 3d becomes $3+1$d automatically, with the tauquernions' irreversibility showing up as global entropy growth.

Applying tauquernions to the construction of relativity theory's 3+1d world of space, time, gravity, mass, and entropy yields a very good fit. Furthermore, the tauquernions are *also*, it turns out, the Bell and Magic entanglement operators of quantum mechanics [8], leading to the conclusion that the mechanism of gravity is the quantum entanglement of space itself.

Further investigation of the mathematical structure of the tauquernions eventually produced the realization that is the tool of this paper: that there is *exactly* <u>one</u> <u>more</u> example of a tauquernion-type operator, namely a *TauQuinion*, $ab+cde$.

Like tauquernions, *tauquinions* are also 4^{th} roots of a projector, $-1 \pm abcde$ in this case, and similarly mutually perpendicular. [So we're now in $\mathscr{G}_5$ over $\mathbb{Z}_3$.] Like tauquernions, the tauquinions are also quaternion isomorphs, and so similarly form an implicitly $3+1$d

[1] How do you want your $\mathbb{Z}_2$ negatives - sign-magnitude (oops, *minus* zero); 1's complement (oops, *two* zeroes), 2's complement (huhhh??) ? It's hard to think of any of these as direct representations of anything physical.

[2] Coldea *et al*: "Our results demonstrate the power of symmetry to describe complex quantum behaviors."

[3] Note that $(ab)^4 = +1$ and $ab = -ba$ together imply that $ab \cong \sqrt{-1}$, the imaginary unit i.

coordinate system - this being the "field" of that for which the projector probes (or, alternatively, the field in which it probes). [We note that for us, a *field* is a coordinate system - which is $\mathscr{G}_5$ itself, it being coordinate-free - having one or more properties assigned to each of its points.]

In the case of the tauquinions, their structure can be reduced to a $3^2 - 1 \Rightarrow 8 \times 8$ group table, and anticipating various other features explained in later sections, we believe that despite apparent differences, the tauquinion group τ is either a representation of $SU(3)$, or contains it. [4]

Appendix I of [7] derives the Standard Model's particle structure almost mechanically from the combinatorics of $\mathscr{G}_3$, which is isomorphic to the Pauli algebra via the mapping $\{iab, iac, ibc\} \mapsto \{\sigma_1, \sigma_2, \sigma_3\}$. This and the smooth way that tauquernions (elements of $\mathscr{G}_4$) build the bridge between QM and GR encourage us to believe that our mathematics connects very well to physics at the level of information.

We show that tauquinions are the *only* other possible tauquernion-like form in $\mathscr{G}_n$. The tauquernion field by itself - which yields $3+1$d spacetime, mass, gravity etc. - does not contain any details beyond a photon $(a+b+c)$ and its local coordinate (ie. $abc(a+b+c) = ab+bc+ca$, a quaternion triple). So where else in the algebra could the electromagnetic field be? Spinors (2-vectors) are clearly to be associated with magnetism, and we identify 3-vectors like abc as the carriers of electric charge [7], which $2+3=5$ structure fits tauquinions and $\mathscr{G}_5$ nicely.

Also, one must not forget that this is a *combinatorial* model over *arbitrary* processes, and so it is *in principle* theory-neutral. If τ does *not* contain $SU(3)$, one is forced to consider that it is $SU(3)$ that is off, which is hugely unlikely. Howsoever, either $SU(3)$ is in τ or it is not, and either answer will be interesting.

A note on terminology. I have always been suspicious of people who make up new words, but it seems justified in the present case. The *tauquernions* $\{ab+cd\}$ are time-like isomorphs of the classical quaternions (which have just one spinor component), and play the starring role in forming $3+1$d spacetime from the quantum mechanical soup [7]. Surely this deserves its own name.

Then some time later, up popped the *tauquinions* $\{ab+cde\}$, quaternion isomorphs with a five-ness (L. *quinque*), that are the star of this paper.

Tauquernions and tauquinions could collectively be called *entangled quaternions*, which indeed they are; and also *entangling*, in that they are also operators. And since both are irreversible (*tau* $\Rightarrow$ time-like) this could be shortened to *enquernions*, in that the Old Norse root of *quern* is *kværn*, meaning to churn ("the churn of time") or grind ("time's tooth"), which is very apt.

0.2 *Notation*

Our mathematics is that of the canonical *geometric (Clifford) algebras* $\mathscr{G}_{n,0} = \mathscr{G}_n$ over $\mathbb{Z}_3 = \{0, 1, -1\}$, whence $1+1=-1$. Such an algebra is generated by a set of 1-vectors $\{a, b, c, ...\}$ with anti-commutative product $xy = -yx$ to produce an orthogonal space of size $|\mathscr{G}_n| = O(2^n)$ with inner and outer products. The usual distributive and associative laws apply. The dimensions of this [coordinate-free] space are all the possible products

[4] We recycle our prior use of τ for tauquernions.

$\binom{\{a,b,c,\ldots\}}{0,1,2,\ldots,n} = \{1,a,b,c,\ldots,ab,ac,\ldots,abc,abd,\ldots,\ldots\}$, which are all mutually orthogonal. See [2,3,4,5] for foundations (although our *interpretation* of the algebra is vastly different).

1-vectors denote discrete processes with two states = 1 bit of information; it follows that an m-vector ("pseudo-vector", "singleton") contains 2^m states. A 1-bit process x is necessarily deterministic, since in each state, there is only the other state to change to. [This is where frequency ν_x attaches.]

In fact, Parseval's Identity [5] applies, so $\mathscr{G}_n$'s discrete datum-space is simultaneously a phase space, namely Fourier space. Every expression in $\mathscr{G}$ is thus the Fourier decomposition of some signal entering the system via the concurrent flipping of some set of 1-vectors at the system boundary.

Generic *concurrent* 1-bit processes x,y (written $x+y$) are considered to be *orthogonal*, which is reflected in the algebra by its anti-commutive product. Time-like sequential processes are represented as products of idempotents and are non-deterministic (as is the entire model). Usually, the expressions we write, eg. $(a+b+ab)^2 = 1$, are valid for all sign variants $\pm a \pm b \pm ab$, each thus constituting a little theorem in itself.

0.3 Roots of Projectors

Projectors (measurement operators) are idempotents, and have the form $\hat{U} = -1 \pm U$, whence $\hat{U}^2 = \hat{U}$ if $U^2 = +1$, ie. if U is unitary. [6] As with scalar +1, the principal square root of $\hat{U}$ is its negative, ie. $\sqrt{\hat{U}} \mapsto -\hat{U}$.

As noted earlier, in our approach $\hat{U}$= "+1" plays a role dual to the scalar $+1$, eg. $(-1+abcde)(ab+cde) = ab+cde$. Likewise, $\sqrt{-\hat{U}} = ab+cde$, being namely an enquernion, plays the role of "i", and $i^2 = -\hat{U} = \sqrt{\hat{U}} = +1 \mp abcde$, which is simultaneously "-1": $(+1-abcde)(ab+cde) = -ab-cde$. This combination of reversible additive inversion ($= 180^o$ rotation) and multiplicative irreversibility is *the* key property of the enquernions, and is the source of their power.

Table 1 displays this roots-of-projectors structure.

scalar	*quaternion*		*time-like*	*tauquernion*	*tauquinion*
$+1$	$+1$	point	$\text{“}+1\text{”} = -1 \pm U$	$-1 \pm abcd$	$-1 \pm abcde$
$\Updownarrow$	$\Updownarrow$		$\Updownarrow$		
$-1 = \sqrt{+1}$	$-1 = \sqrt{+1}$	line±	$\text{“}-1\text{”} = \sqrt{\text{“}+1\text{”}}$	$+1 \mp abcd$	$+1 \mp abcde$
$\Updownarrow$	$\Updownarrow$		$\Updownarrow$		
$i = \pm\sqrt{-1}$	$ab = \pm\sqrt{-1}$	plane(s)	$\text{“}i\text{”} = \pm\sqrt{\text{“}-1\text{”}}$	$ab+cd$	$ab+cde$

TABLE 1. Idempotent Isomorphs of $+1$ and $\hat{U} = -1 \pm U$, $U^2 = +1$.

The first column is the usual $i = \sqrt{-1}$ story. This same vertical progression, in a vector space, yields a spinor/quaternion ab (second column). A 3d rotation in $\mathscr{G}_3$, $abc(a+b+c) = ab+bc+ca$, demonstrates the equivalence of a quaternion triple to a 3d space with axes a,b,c. Like a,b,c, the three quaternion elements anti-commute.

[5] " Given an orthogonal space S with inner product, the projection of any function $\mathscr{F}$ onto S is the Fourier decomposition of $\mathscr{F}$ ". The Identity is a generalization of the Pythagorean theorem to n dimensions. It is also, for us, both wave-particle duality in a nutshell, and the bridge between the discrete and the continuous.

[6] In $\mathbb{Z}_3$: the -1 in $\hat{U}$ corresponds to decimal $\frac{1}{2}$, which latter one finds in the base-10 version of a projector.

The third column indicates the progression of geometric concepts that appears in this process, and also divides the table in half, in that the i's to the left are reversible, ie. have multiplicative inverses, while those to the right do not, thus making them time-like (in addition to being space-like) rotation operators. The enquernion i's to the right are, spatially speaking, isomorphic to the quaternions' i's (as we will demonstrate later), but provide a new and unique twist, namely that even though having no inverse in $\mathscr{G}_5$, they yet perform the requisite anti-commutative space-like rotations that make them quaternion isomorphs.

Another way to look at enquernions is that they connect a quantum level change to an *exactly* equivalent $3+1$d change, via the identity

$$(-1)(ab \pm cde) = (1 \mp abcde)(ab \pm cde)$$

On the left is a reversible change, on the right an irreversible one.

It seems therefore that the roots of projectors can specify the formation, structure and state of the field associated with the unitary entity $abcde$. We pursue this thought in §2.

0.4 *Hierarchy*

The algebra has the recursive (and hence hierarchical) property, that its semantics cycle exactly like the powers of i : $\{++--++--\dots\}$ as the grade of its (pseudo-)vectors I increases:

grade	$I \in \mathscr{G}_n$	*notation*	I^2
...	...	...	...
10	...	A_{10}	-1
9	$abcdefghj$	A_9	$+1$
8	$abcdefgh$	A_8	$+1$
7	$abcdefg$	A_7	-1
6	$abcdef$	A_6	-1
5	$abcde$	A_5	$+1$
4	$abcd$	A_4	$+1$
3	abc	A_3	-1
2	ab	A_2	-1
1	a	A_1	$+1$
0	1	A_0	$+1$

Note particularly that the sequence repeats every four levels, so (eg.) m-vectors of grades 1 and 5 have identical properties. This we exploit in the following, as the means by which a field propagates from its initial locality to a global presence, all the while and everywhere retaining its defining local properties. [7]

Finally, we use the boundary ∂ and co-boundary δ operators to define and build the algebra's graded hierarchy of m-vectors. There is a deep analogy between ∂ as a boundary operator and the differentiation operation of the calculus, and similarly between the co-boundary operator δ and integration.

We take ∂ and δ to be elements *of* the algebra - rather than the usual operators *over* the algebra - this being a less sophisticated but more concrete encoding of the same ideas, thus the definition

$$\delta_Q X = \pm Q \quad \textit{iff} \quad \partial_X Q = XQ \quad \textit{and} \quad XQ \cong X$$

[7] There is also a *mod* 8 cycle, over the kinds of spaces spanned, tangential to our present purposes.

This definition specifies that a boundary X of Q must be an *eigen-form* of Q for the analogy to hold, and it is not difficult to show that it requisitely satisfies $\partial\delta = 1 = \delta\partial$ and $\partial^j = 0 = \delta^k$. We use δ to build hierarchy, eg. $\delta_{ab}(a+b) = ab$, since $\partial_{a+b}ab = (a+b)ab = -a+b \cong a+b$.

See [7] for more.

1. The TauQuinions are isomorphic to the classical quaternions.

The classical quaternions have the following multiplication table:[8]

$\times$	$Q_i = ab$	$Q_j = ac$	$Q_k = bc$
Q_i	-1	$-bc$	ac
Q_j	bc	-1	$-ab$
Q_k	$-ac$	ab	-1

=

$\times$	Q_i	Q_j	Q_k
Q_i	-1	$-Q_k$	Q_j
Q_j	Q_k	-1	$-Q_i$
Q_k	$-Q_j$	Q_i	-1

The respective tauquinions are $\tau_i = ab+cde$, $\tau_j = ac-bde$, $\tau_k = bc+ade$. Their multiplication table is below left; below right is the same table, but with the mapping $1-abcde \mapsto$ "-1".

$\times$	$\tau_i = ab+cde$	$\tau_j = ac-bde$	$\tau_k = bc+ade$
τ_i	$1-abcde$	$bc+ade$	$-ac+bde$
τ_j	$-bc-ade$	$1-abcde$	$ab+cde$
τ_k	$ac-bde$	$-ab-cde$	$1-abcde$

=

$\times$	τ_i	τ_j	τ_k
τ_i	"-1"	-τ_k	τ_j
τ_j	τ_k	"-1"	-τ_i
τ_k	-τ_j	τ_i	"-1"

Like the Q's, the τ's anti-commute, eg. $\tau_i\tau_j$= -$\tau_j\tau_i$; close circularly, eg. $\tau_i\tau_k$=τ_j; and -$\tau_i\tau_j\tau_k$ = $\tau_k\tau_j\tau_i$. We emphasize that these tauquinion relationships are independent of the restriction to $\mathbb{Z}_3$. Clearly, the two tables to the right, quaternion and tauquinion, are isomorphic. So if you can build a field with quaternions, you can build it with tauquinions too.

There are $\binom{5}{2} = \binom{5}{3} = \frac{5\cdot4}{2} = 10$ such tauquinion pairs in $\mathscr{G}_5$: [9]

$$\{ab+cde, ac+bde, ad+bce, ae+bcd, bc+ade,\\ bd+ace, be+acd, cd+abe, ce+abd, de+abc\}$$

Conjugates to these form another (dual) space

$$\{ab-cde, ac-bde, ad-bce, ae-bcd, bc-ade,\\ bd-ace, be-acd, cd-abe, ce-abd, de-abc\}$$

Including negatives, there are 40 tauquinion pairs in all. Most of these form a quaternion triplet with two others, with some products producing, however, some form of -1. [10] A full product table appears later.

[8] Here and and later, we use a collapsed form of group table, where the redundancies of "times +1", "times -1", and anti-commutation are not listed. Full 8x8 tables appear in the second Appendix.

[9] Versus tauquernions: $\binom{4}{2} = 6$, and the disjointness criterion then produces three tauquernion pairs.

[10] Choosing a particular triple (there are 32 in the 10×10 table given later) constitutes an arbitrary choice of coordinate system orientation, cf. the "right hand rule" in 3d.

2. The uniqueness of tauquernion $\{ab+cd\}$ ***and tauquinion*** $\{ab+cde\}$ ***forms.***

Let X,Y be two pseudo-vectors, whence $X^2=Y^2=\pm 1$. We wish to determine the conditions under which

$$(X+Y)^2 \;=\; \text{"}-1\text{"} \;=\; 1\mp XY \;=\; -\sqrt{\text{“}+1\text{”}} \;=\; -\sqrt{-1\pm XY}$$

where $-1\pm XY=\text{"}+1\text{"}$ is a projector with unitary element XY. If these be so, then $X+Y$ is an analog to $\sqrt{-1}$, and hence can perform the $4\times\frac{\pi}{2}$ rotations that $i=\sqrt{-1}$ performs. This in turn implicitly gives two orthogonal dimensions, from which we can then construct a third, *a la* classical quaternions.

Achieving such a $\sqrt{-1}$ analog requires both that $(XY)^2=+1$ and $(X+Y)^2=\text{“}-1\text{”}$ as above. Since $(XY)^2=+1$, XY must have grade $\{0,1\}\,mod\,4$, cf. preceding table; and as well, satisfy $\delta(X+Y)=XY$, which means that X and Y must be disjoint. Re $+1\mp XY$,

$$(X+Y)(X+Y) \;=\; X^2+XY+YX+Y^2 \;=\; \pm 1 \,+\, (XY+YX) \,\pm\, 1$$

so to get the desired unitary XY, X and Y must commute, the right-most yielding

$$=\; \pm 1 \,-\, XY \,\pm\, 1$$

since $XY+XY=-XY$ in $\mathbb{Z}_3$. This commutativity means that at least one of X,Y has grade $\{0,2\}\,mod\,4$. Say $X_2=ab$. Then Y can be of either even or odd grade.

To get a $+1$ (in "-1" $=+1\pm XY$), both X and Y must square to -1, which means that X,Y must both have grade $\{2,3\}\,mod\,4$. So $X_2=ab$ still holds.

If we then choose Y to also be of *even* grade, $Y_2=cd$, we get the tauquernion family $\{ab+cd\}$. Choosing X,Y to be both of grade 4 yields $(X_4+Y_4)^2=(-1\pm X_4Y_4)$, which is "$+1$", not "$-1$". Choosing grades 2 and 4 yields grade 6, whence $(X_2Y_4)^2=-1$, not $+1$. But since $6=2\,mod\,4$, the base case is $2+2=4$, the tauquernions. [11]

Instead, if for *odd* Y we choose grade $=1$, then $Y_1=c$ and we would have $X_2+Y_1=ab+c$, which leads to the quark family [7]; so we choose grade = 3, say $Y_3=cde$. This then yields the tauquinion family $\{ab+cde\}$. Choosing $odd=1$ and $even=4$ is in the tauquinion table automatically - see below. So the base case is $2+3$, the tauquinions.

Thus, due to the algebra's $mod\,4$ cycle, any disjoint pair with the grade structure $2\,mod\,4+3\,mod\,4$ will have tauquinion properties. Similarly, any pair with the grade structure $2\,mod\,4+2\,mod\,4$ will have tauquernion properties. But no others are possible, since the tauquernion and tauquinion forms exhaust the algebra's possibilities in this regard.

Note that $1+3=4$ appears nowhere. Cf. [7], this corresponds to dark matter. Because $(1\,mod\,4+3\,mod\,4)^2$ is always zero, yielding no “$\pm 1\pm U$” at all, dark matter is therefore out of the structure game except as it is associated with tauquernions. We return to this later.

Summarizing, there are then *just these two groups*, tauquernions and tauquinions, of fundamental field-generators derived from pairs of pseudo-vectors $X+Y$ such that $(X+Y)^2=$ “-1”, which (“-1”$)^2$ is the idempotent operator “$+1$”$=\hat{U}=-1\pm XY$.

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[11] Note that $6+6=12=0\,mod\,4$ works too. Tauquernion space (like tauquinion space, as we shall see) thus expands in two ways, $4+2=6=2\,mod\,4 \rightsquigarrow 6+6=12=0\,mod\,4$ and $1+1=2\rightsquigarrow 2+2=4=0\,mod\,4$. But we keep our focus on tauquinion space.

3. How the tauquinion field is organized and propagated.

Both the tauquernion and tauquinion fields achieve their coverage in two ways:

(1) Lateral recombination at/across the same two grade-levels, eg. $(ab+cde)+(fg+hij) \rightsquigarrow (ab+hij)+(fg+cde)$. Note that this recombination is $\mathscr{O}(n^2)$, and so convenient to associate a dilution effect of $\mathscr{O}(\frac{1}{r^2})$ [once the tauquernions have built $3+1$d]. { "$\rightsquigarrow$" means "leads to"}

(2) Hierarchical consolidation, eg. $\delta(ab+cde) = abcde$.

The latter, which models local-to-global propagation (vs. Lateral's local-to-local), occurs via the co-boundary sequence (from bottom row, upwards)

pairs	$\delta(pair)$	*new level*		
$3\,mod\,4+3\,mod\,4$	$\rightsquigarrow$	6	$=2\,mod\,4$	$\searrow$
$2\,mod\,4+3\,mod\,4$	$\rightsquigarrow$	5	$=1\,mod\,4$	$\searrow$
$2\,mod\,4+2\,mod\,4$	$\rightsquigarrow$	4	$=0\,mod\,4$	$\searrow$
$1\,mod\,4+2\,mod\,4$	$\rightsquigarrow$	3	*charge*	. $\downarrow$
$1\,mod\,4+1\,mod\,4$	$\rightsquigarrow$	2	*spin*	$\overset{6}{\leftarrow}\downarrow$
$0\,mod\,4+1\,mod\,4$	$\rightsquigarrow$	1	*existence*	$\overset{4,5}{\leftarrow}\downarrow$

wherein we see the formative role to be played by the algebra's telescoping of its semantic levels $mod\,4$ (right-most column). In particular, the $2+3=5 \overset{mod4}{\longrightarrow} 1$ loop propagates its form in every cycle of hierarchical consolidation, always being created from previous levels $\{2,3\}\,mod\,4$. In this way a field is propagated "up" to macroscopic size, whence its basic character is the same at all scales.

The table below displays, in the left-most six columns, the algebraic buildup of higher and higher grade pseudo-vectors in this $2+3=5 \mapsto 1$ fashion.

pair	$\delta(pair)$	*pair*	$\delta(pair)$	*pair*	$\delta(pair)$	*grade g*
				$\dots$	$\dots$	$\dots$
				C_2+C_3	C_5	50+75=**125**
				C_2+C_2	C_4	50+50=**100**
				C_1+C_2	C_3	25+50=**75**
				C_1+C_1	C_2	25+25=**50**
		B_1+B_4	B_5	$\mapsto C_1$		5+20=**25**
		B_2+B_3	B_5	$\mapsto C_1$		10+15=**25**
		B_2+B_2	B_4			10+10=**20**
		B_1+B_2	B_3			5+10=**15**
		B_1+B_1	B_2			5+5=**10**
A_1+A_4	A_5	$\mapsto B_1$				1+4=**5**
A_2+A_3	A_5	$\mapsto B_1$				2+3=**5**
A_2+A_2	A_4					2+2=**4**
A_1+A_2	A_3					2+1=**3**
A_1+A_1	A_2					1+1=**2**
A_1						0+1=**1**
A_1						**1**
A_0						**0**

The rightmost column, *grade g*, is the grade of the *m*-vector created by $\delta(pair)$.

pair	*pair*	*pair*	*grade g*	*mod* 4	*i*	F_i	F_i *mod* 4
		...	...	**0**	***12***	144	**0**
		C_2+C_3	50+75=125	**1**	***11***	89	**1**
		C_2+C_2	50+50=100				
		C_1+C_2	25+50=75	**3**	***10***	55	**3**
		C_1+C_1	25+25=50	**2**	***9***	34	**2**
	B_1+B_4	$\mapsto C_1$	5+20=25				
	B_2+B_3	$\mapsto C_1$	10+15=25	**1**	***8***	21	**1**
	B_2+B_2		10+10=20				
	B_1+B_2		5+10=15	**3**	***7***	13	**1**
	B_1+B_1		5+5=10	**2**	***6***	8	**0**
A_1+A_4	$\mapsto B_1$		1+4=5				
A_2+A_3	$\mapsto B_1$		2+3=5	**1**	***5***	5	**1**
A_2+A_2			2+2=4				
A_1+A_2			2+1=3	**3**	***4***	3	**3**
A_1+A_1			1+1=2	**2**	***3***	2	**2**
A_1			0+1=1	**1**	***2***	1	**1**
A_1			1	**1**	***1***	1	**1**
A_0			0	**0**	***0***	0	**0**

Note now the next column [new, above], labelled *mod* 4, ie. $g\,mod\,4$. For example, the grade of the *m*-vector made by $\delta(A_2+A_3) = A_5 \mapsto B_1$ is the sum of the grades of its two constituents, namely $2+3=5$, and $5\,mod\,4 = 1$; and similarly the next octave up, $B_2+B_3 \mapsto 10+15 = 25$, we get C_1, and $25\,mod\,4 = 1$. So $B_2+B_3 \mapsto C_1$ is just like $A_2+A_3 \mapsto B_1$.

The last three columns are, respectively, a counter *i*, the corresponding element F_i of the Fibonacci series, and finally $F_i\,mod\,4$. Note that the two *mod* 4 columns correspond closely; see the footnote *re* the differences at steps 6 and 7. [12] Furthermore, the ratio of successive entries in the *grade g* column approximates the golden mean φ, eg. $g_{14}/g_{13} = 525/325 = 1.6153\ldots$ *vs.* $\varphi = (1+\sqrt{5})/2 = 1.6180\ldots$, as one would expect. That is, the tauquinion field has an underlying deep Fibonacci structure! [13] [14]

It is at this level of abstraction that we connect to Coldea *et al*'s finding [1].

The $\mathbb{Z}_3\ \mathscr{G}_n$ picture is a bit like an x-ray photograph of the quantum mechanical world - it shows the overall bone structure, but the details of the flesh - of which there are *many* - must be provided from elsewhere (ie. known physics). In fact, one could argue that our analysis shows that much of the mathematical thicket that is quantum mechanics is (apparently) concerned with expressing process, structure, and their inter-relationship. Because in our algebra, vectors are processes and structure derives from their co-occurrence ("+"), these two are completely integrated, and the algebra's minimality wraps the physics tightly.

[12] The differences originate in $5+5 = (1+1)+(4+4) = 10 = 2\,mod\,4$ versus F's $3+5 = 8 = 0\,mod\,4$. Since $4+4 = 0\,mod\,4$, the two differ by $1+1 = 2$; step 7 then adds 1 to get 3. The two extra 1's ultimately stem from δ's *doubling* of 5 (vs. doubling 4). The two sequences are identical $mod\,2 \rightsquigarrow (0,1,1)^{1*}$, and hence rejoin explicitly every other F-cycle, next at $F_6+F_7 = F_8$. [^{1*} means one or more instances]

[13] Actually, *any* such accumulating sequence with positive numbers will converge to φ, but the subject of §5 shows that this version of the sequence has its origin in Fibonacci's.

[14] As with tauquernions, there are two tauquinion sequences: $2+3 = 5 \mapsto 1$ and $3+3 = 6 \rightsquigarrow 6+6 = 12 \mapsto 0$. I speculate that the paired Fibonacci numbers in (eg.) pinecones and sunflowers arise from this.

Stepping back, it is apparent that the emergence of the Fibonacci sequence at the quantum level is a mathematical inevitability that Coldea et al.'s experimental result confirms. It is the dual physical *and* computational, process-oriented interpretations that we *simultaneously* lay on the algebra that give this conclusion conceptual heft.

Herein also lies a prediction, since the τ sequence does differ slightly ($mod\,4$ vs. $mod\,2$) from the mathematical ideal, which might perhaps be measurable. This measurement would tell us if the universe is fundamentally built on the rationals (the $mod\,4$ sequence) or the reals (the $mod\,2$ sequence, which approximates the irrational φ better).

Furthermore, the correspondence we have found is not just the usual numerical sequence - it is also, uniquely, *exact operators and actual states*, a gift from $\mathscr{G}$'s graded structure. That is, one would expect that the Fibonacci properties of macroscopic entities like flowers, pinecones, and sea shells are brought about by the operation of tauquinions, steering the growth. As well, since electro-magnetism itself is very well characterized, one can inquire directly. [15]

We needn't restrict $F_i\,mod\,n$ to $n = 4$: any n produces a pattern *a la* $(011231)^{1*}$ above, but more jumbled (ie. longer); $mod\,2$ produces $(011)^{1*}$. Indeed, the fact that these patterns are all (more and less) simultaneously present means that the hierarchy, and the patterns themselves, are fundamentally fractal in nature.

It is interesting to see what the significant neighbor to Fibonacci's sequence, namely Lucas' sequence, says:

$$L_n = F_{n+1} + F_{n-1} = (F_n + F_{n-1}) + F_{n-1}$$

i	L_i	$(L_{i-2} + L_{i-1})\,mod\,4$
11	123	$3+0 \mapsto 3$
10	76	$1+3 \mapsto 0$
9	47	$2+1 \mapsto 3$
8	29	$3+2 \mapsto 1$
7	18	$3+3 \mapsto 2$
6	11	$0+3 \mapsto 3$
5	7	$3+0 \mapsto 3$
4	4	$1+3 \mapsto 0$
3	3	$2+1 \mapsto 3$
2	1	1
1	2	2

As noted earlier, we identify the form $w + xyz =$ "$1 + 3$" as dark matter. This form is nilpotent in all sign variants, and so cannot be the "-1"-like root of a projector $-1 \pm wxyz$ that we seek, even though $\delta(w + xyz) = wxyz$. Therefore, it cannot be the basis for a coordinate system.

Thus dark matter is no part of the tauquinion field, nor its putative electro-magnetic properties.

[15] Even though the Fibonacci imprint is present from the very bottom of the hierarchy, this same mechanism's effect or appearance at higher levels is not reductionistically dependent on this prior presence. Rather, the same forms can emerge spontaneously at any given level as a collective (ie. emergent) property of the very [growth!] processes that are taking place there. Maybe Fibonacci cities are the future [9].

4. A Closer Look at the TauQuinion Field

Here is a full tauquinion set of ten, chosen such that their sum is nilpotent, ie. field-like. NB: In the table below, "..−1.." = $+1-abcde$:

$\times$	$ab+cde$	$ac-bde$	$ad+bce$	$ae-bcd$	$bc+ade$	$bd-ace$	$be+acd$	$cd+abe$	$ce-abd$	$de+abc$
$ab+cde$	..−1..	$-bc-ade$	$-bd+ace$	$-be-acd$	$ac-bde$	$ad+bce$	$ae-bcd$	$e-abcd$	$-d-abce$	$c-abde$
$ac-bde$	$bc+ade$	..−1..	$-cd-abe$	$-ce+abd$	$-ab-cde$	$-e+abcd$	$d+abce$	$ad+bce$	$ae-bcd$	$-b-acde$
$ad+bce$	$bd-ace$	$cd+abe$	..−1..	$-de-abc$	$e-abcd$	$-ab-cde$	$-c+abde$	$-ac+bde$	$b+acde$	$ae-bcd$
$ae-bcd$	$be+acd$	$ce-abd$	$de+abc$	..−1..	$-d-abce$	$c-abde$	$-ab-cde$	$-b-acde$	$-ac+bde$	$-ad-bce$
$bc+ade$	$-ac+bde$	$ab+cde$	$e-abcd$	$-d-abce$	..−1..	$-cd-abe$	$-ce+abd$	$bd-ace$	$be+acd$	$a-bcde$
$bd-ace$	$-ad-bce$	$-e+abcd$	$ab+cde$	$c-abde$	$cd+abe$	..−1..	$-de-abc$	$-bc-ade$	$-a+bcde$	$be+acd$
$be+acd$	$-ae+bcd$	$d+abce$	$-c+abde$	$ab+cde$	$ce-abd$	$de+abc$	..−1..	$a-bcde$	$-bc-ade$	$-bd+ace$
$cd+abe$	$e-abcd$	$-ad-bce$	$ac-bde$	$-b-acde$	$-bd+ace$	$bc+ade$	$a-bcde$	..−1..	$-de-abc$	$ce-abd$
$ce-abd$	$-d-abce$	$-ae+bcd$	$b+acde$	$ac-bde$	$-be-acd$	$-a+bcde$	$bc+ade$	$de+abc$	..−1..	$-cd-abe$
$de+abc$	$c-abde$	$-b-acde$	$-ae+bcd$	$ad+bce$	$a-bcde$	$-be-acd$	$bd-ace$	$-ce+abd$	$cd+abe$	..−1..

Note that there are five groups, defined by their particular $v+wxyz=$ '-1'. To focus on one of these groups, taking only those 2-vectors that belong to $\mathscr{G}_4$ on$\{a,b,c,d\}$ reduces this 10×10 table to 6×6:

$\times$	$ab+cde$	$ac-bde$	$ad+bce$	$-bc-ade$	$bd-ace$	$-cd-abe$
$ab+cde$	$1-abcde$	$-bc-ade$	$-bd+ace$	$-ac+bde$	$ad+bce$	$-e+abcd$
$ac-bde$	$bc+ade$	$1-abcde$	$-cd-abe$	$ab+cde$	$-e+abcd$	$-ad-bce$
$ad+bce$	$bd-ace$	$cd+abe$	$1-abcde$	$-e+abcd$	$-ab-cde$	$ac-bde$
$-bc-ade$	$ac-bde$	$-ab-cde$	$-e+abcd$	$1-abcde$	$cd+abe$	$bd-ace$
$bd-ace$	$-ad-bce$	$-e+abcd$	$ab+cde$	$-cd-abe$	$1-abcde$	$bc+ade$
$-cd-abe$	$-e+abcd$	$ad+bce$	$-ac+bde$	$-bd+ace$	$-bc-ade$	$1-abcde$

This particular set of tauquinions was chosen so that the 2-vectors form a Higgs boson $\mathscr{H}$ (ie. nilpotent). The 3-vector electric component $\mathscr{E}$ is also nilpotent, as is $\mathscr{H}+\mathscr{E}$. [16]

Noting that *both* $1-abcde =$ "-1" *and* $-e+abcd =$ '-1', which we take to be the magnetic and electrical polarity indicators, respectively, the preceding table can be rewritten a bit more clearly:

$\times$	$ab+cde$	$ac-bde$	$ad+bce$	$-bc-ade$	$bd-ace$	$-cd-abe$
$ab+cde$	"-1"	$-bc-ade$	$-bd+ace$	$-ac+bde$	$ad+bce$	'-1'
$ac-bde$	$bc+ade$	"-1"	$-cd-abe$	$ab+cde$	'-1'	$-ad-bce$
$ad+bce$	$bd-ace$	$cd+abe$	"-1"	'-1'	-$ab-cde$	$ac-bde$
$-bc-ade$	$ac-bde$	$-ab-cde$	'-1'	"-1"	$cd+abe$	$bd-ace$
$bd-ace$	$-ad-bce$	'-1'	$ab+cde$	$-cd-abe$	"-1"	$bc+ade$
$-cd-abe$	'-1'	$ad+bce$	$-ac+bde$	$-bd+ace$	$-bc-ade$	"-1"

This table is then, presumably, the entire field situation in 3-space at a single point, as specified by the tauquernion subset, thus "locating" the electro-magnetic field, which is then further specified by the tauquinion relationships, which latter are thus automatically constrained in $3+1$d by their tauquernion components.

An electro-magnetic field interplays two polarities - magnetic and electric - which polarities are ultimately specified by the orientations of the associated spinors. The 2-spinor configuration defines the *magnetic* field in 3-space, and its minus-sign is indicated by the NW-SE diagonal, "-1"$= 1\pm abcde$.

Similarly, the 3-spinor configuration defines the *electric* field $\mathscr{E}$, and its polar indicator '-1'$=\pm(e-abcd)$ is the NE-SW diagonal; note that electric-plus is $(e\pm abcd)^2 = -1\pm abcde =$ "$+1$" as it should. However, inversion's '-1' is "minus" with a twist: $(e-abcd)(ab+cde) = -cd-abe$, so not only are both charges reversed, but the electric and magnetic components do a dosey-do as well; fans of Maxwell's equations will recognize this as the mechanism of induction. More straightforwardly, $(1-abcde)(-e+abcd) = e-abcd$. A photon $a+b+c$ interacting with the field looks like

$$(a+b+c)(de+abc) = (ab+cde)-(ac-bde)+(bc+ade)$$

where these three tauqinion terms define the field orientations associated with the photon. Re-arranging the rhs yields

$$= (ab-ac+bc)+(a+b+c)de$$

which are the magnetic and electric field orientations, respectively, taken separately. Note that $a+b+c$ is a 1-vector with a definite direction, and that the electric field is therefore a de-rotation relative to (ie. perpendicular to) this direction.

[16] Such nilpotent forms are a real minority - only 240, versus 13,200 that are not.

The following relationships hold for the above-specified field. However, due to the extreme symmetry of $\mathscr{G}_n$ over $\mathbb{Z}_3$, one can view them as true for all field-type states; other states are roots of unity.

Tauquernions $\tau_i = ab - cd$; $\tau_j = ac + bd$; $\tau_k = ad - bc$. $\mathscr{H}$ is the Higgs boson.

$$\mathscr{H} = \tau_i + \tau_j + \tau_k \qquad \mathscr{H} * \mathscr{H} = 0$$

$$\mathscr{E} = cde - bde + bce - ade - ace - abe \qquad \mathscr{E} * \mathscr{E} = 0$$

$$= (cd - bd + bc - ad - ac - ab)e = -\mathscr{H}e$$

$$\mathscr{H} + \mathscr{E} = \mathscr{H} - \mathscr{H}e = \mathscr{H}(1-e) \qquad (\mathscr{E} + \mathscr{H})^2 = 0$$

Since $\mathscr{E} = -\mathscr{H}e$, $\mathscr{E}$ is compatibly entangled with the gravitational field formed by $\mathscr{H}$.

Finally, Appendix I describes the operators that generate Majorana particles, which turn out to be enquernions. Majorana states are of great interest in both quantum computing and quantum theory, and have recently (possibly) been observed [11].

5. A Final Puzzle

If one lists all the unitary elements U in the algebra $\mathscr{G}_3$ (isomorphic to the Pauli algebra), one finds the following ($U^2 = +1$):

$$1, \;\; a, \quad ab + ac, \quad a + b + ab, \quad a + b + c + ab + ac$$

with dimensionalities (= terms) respectively 1,1,2,3,5. Until now, it has been entirely opaque as to whether this was chance or a (maybe) instance of a Fibonacci progression. Now, the problem is reversed: how to understand these given the preceding analysis.

The unitary elements U we have discussed in earlier sections have always been a singleton term, and these constructed via δ from pairs of same. Now, however, our idempotents look like

$$-1 + a + b + ab, \quad -1 + ab + ac, \quad -1 + a + b + c + ab + ac$$

Where do these multi-term U's come from? Our usual constructor, the co-boundary operator δ, fails. Adding suspense to the story, these $-1+U$'s we have elsewhere identified as the neutrino, electron, and proton projectors, respectively [7]. Do they nevertheless have the roots we require? [see Table 2.]

time-like	neutrino	electron	proton
"$+1$" $= -1 \pm U$	$-1+a+b+ab$	$-1+ab+ac$	$-1+a+b+c+ab+ac$
$\Updownarrow$			
"-1" $= \sqrt{\text{"}+1\text{"}}$	$1-a-b-ab$	$1-ab-ac$	$1-a-b-c-ab-ac$
$\Updownarrow$			
"i" $= -\sqrt{\text{"}-1\text{"}}$	none	$\pm(b-c-abc)$ $= abcU$	$\pm(b-c+ab-ac+bc-abc)$ $= abcU$; $(b-c+ab-ac+bc)^4 = U$

TABLE 2. Time-like roots of stable particles.

Clearly, from the table, we can try to apply our same reasoning with electrons and protons - they at least *have* fourth roots - but neutrinos will need a different treatment (next ¶+1). If we are to remain faithful to our earlier interpretation of $\mathscr{H}$ and $\mathscr{E}$ as field generators, then $b - c - abc$ too should be the actual field element associated with the electron projector

$-1+ab+ac$, and namely charged: $-abc$; and similarly for the proton. Certainly, $b-c-abc$ is, like an electron, very nearly a geometric point: an oriented volume $-abc$ with mostly missing sides ($b-c$ only determines a plane bc); the proton's fourth root is similarly missing various faces.

Do the multiple i's (for a given particle form e or p) anti-commute with each other *a la* quaternions and enquernions? Ie. make tiny $3+1$d spaces that tile up into something macroscopic? No, they remain isolated. Instead of anti-commuting, these products produce each other's additive inverse. That is, they do not form quaternion triples, and so no field. In fact, these fourth root "imaginaries" are just abc rotations of the original idempotent form, not at all what is needed.

Regarding the neutrino, the form $\pm 1 \pm a \pm b \pm ab$ always factors into one of the products $(\pm 1 \pm a)(\pm 1 \pm b)$ or $(\pm 1 \pm b)(\pm 1 \pm a)$, of which there are sixteen in all. Those with -1 are idempotents, and as before, their negatives (ie. with $+1$) are their square roots (sqerts). But alas, the neutrino's sqerts themselves have no square roots at all, so our game is stymied again.

Despite the fact that the neutrino's structural dynamic differs from that of electrons and protons, which themselves aren't simple pseudo-vectors either, *some* juggling act nevertheless ends up creating the first five members of the Fibonacci sequence, even though what's going on is, by definition, completely uncoordinated co-occurrences of small 1- and 2-vector "atoms" ... in an entirely non-deterministic process frenzy that is nevertheless self-synchronized and convergent ... to namely $1,1,2,3,5$. How might this come to be?

The three complex oscillations - engendered ultimately by the Bit Bang's entropy creation [7] - are stable because they are unitary (and entropically favored). Their structures are inter-connected:

	ab		~~ac~~	
a	+	b		~~c~~

	ab	+	ac	
~~a~~		~~b~~		~~c~~

	ab	+	ac	
a	+	b	+	c

The transition $a+b \overset{\delta}{\longmapsto} ab$ results in the unitary entity $a+b+ab$, which is the simplest possible non-trivial stable oscillatory structure, simplest because it derives from the simplest possible structure-generating distinction: two 1-bit states that co-occur/exclude. $a+c \overset{\delta}{\longmapsto} ac$ proceeds similarly. The co-occurrences $ab+ac$ and $a+b+c+ab+ac$ are unitary already, and do not engender a co-boundary transition.

Their behavior is wave-like, and as the independent elements (the 1-vectors a,b,c) change, so will the spins of ab,ac oscillate accordingly. Note that while particular frequencies ν_a, ν_b, ν_c can be directly associated with a,b,c, the unitarity of $a+b+ab$ and the others is namely *not* dependent on their values.

In conclusion, the actual unitaries $\{ab+ac,\ a+b+ab,\ a+b+c+ab+ac\}$ are the very first generation of the underlying recursive Fibonacci structure, born in the inevitable unitarity of their oscillation.

So why is the Fibonacci sequence the convergent and not something else? Our answer is that the uniqueness of the two enquernion forms, along with the algebra's *mod* 4 cycle, allow very little room for Nature to experiment in. If there are other possible Fibonacci-like sequences, they apparently all fizzle out, eg. the closely related Lucas' sequence, leaving the Fibonacci sequence the only surviving possibility.

One last, mathematical, remark: the golden ratio derives from the proportion $\frac{1}{\varphi} = \frac{\varphi}{1+\varphi}$, leading to the roots (golden ratios) $\varphi = \frac{1+\sqrt{5}}{2} = 1.618033989$ and $\varphi' = \frac{1-\sqrt{5}}{2} = -0.618033989$. It is easy to show that $\varphi = -\frac{1}{\varphi'}$ and vice versa. Consider now the following (left-to-right): [Recall that $U = U^{-1}$ when $U^2 = +1$]

$$\varphi = -\frac{1}{\varphi'} \qquad i = -\frac{1}{i} \qquad \tau_i = \text{“}-\frac{1}{\tau_i}\text{”} = -(\tau_j\tau_k)^{-1} = -\tau_k\tau_j$$

or

$$\varphi\varphi' = -1 \qquad i\,i = -1 \qquad \tau_i\tau_i = \tau_i(\tau_j\tau_k) = \text{“}-1\text{”}$$

That is, the multiplicative inverse of φ includes a sign inversion ... just like i ... just like tauquinions. That is, the golden ratios φ and φ', *and* $i = \sqrt{-1}$, *and* $\tau = ab + cde$, *and* $\tau = ab + cd$, are actually all in the same family: fourth roots of unity (ie. of its particular version of "$+1$"). So it is not quite so surprising that the Fibonacci sequence should show up in our analysis of tauquinions. [They are also instances of Jacobian theta/modular functions.]

6. Re *Coldea* et al.*'s findings.*

Coldea *et al.*'s theoretical and experimental considerations lie well outside our expertise. This, we think, supports our case, since we have shown in the preceding that the build-up of structure up in $\mathscr{G}_3$ (= the Standard Model) inevitably produces the Fibonacci sequence, in ignorance of their work.

Considering the $\mathscr{G}_3$ build-up to be the base case in an inductive proof, we then proceeded to show that the $2+3=5 \mapsto 1$ pattern continues into higher grades without bound. Had we not discovered Coldea *et al.'s* result via literature search, we would have predicted that the golden mean would be found in quantum systems, from the simplest to the most complex, and indeed, beyond.

As regards the exceptional Lie group E_8, we do not at this point know if it is present here, or whether we have solved its problem without it.

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References.

1. R. Coldea, *et al*. "Quantum Criticality in an Ising Chain: Experimental Evidence for Emergent E_8 Symmetry." Science 327, 177 (2010); DOI: 10.1126/science.1180085

2. Doran, C. and Lasenby, A. *"Geometric Algebra for Physicists*". Cambridge University Press, 2003. Isbn 978-0-521-71595-9.

3. Hestenes, D. *"New Foundations for Classical Mechanics*" (2^{nd} ed). Kluwer Academic Press, 1999. Isbn 0-7923-5514-8.

4. Hestenes, D. and Sobczyk, G. "*Clifford Algebra to Geometric Calculus - A Unified Language for Mathematics and Physics"*. Kluwer Academic Press, 1984/87. Isbn 90-277-2561-6.

5. Lounesto, P. "*Clifford Algebras and Spinors*". London Mathematical Society, Lecture Notes Series #286. Cambridge University Press (1997) 2002. Isbn 0-521-00551-5.

6. Manthey, M. "Synchronization - the Font of Physical Structure". International Journal of Anticipatory Systems, Daniel Dubois, Ed. Issn 1373 - 5411, v. 20-21-22, IJCAS/BCS Symposium, Liege, Belgium. August 6-11, 2007. Isbn 2-930396-09-1. *Tauquernions.org*

7. Manthey, M. and Matzke, D. "TauQuernions τ_i, τ_j, τ_k: *3+1*d Dissipative Space out of Quantum Mechanics*". Tauquernions.org April 2013*.

8. Matzke, D. *Quantum Computation Using Clifford Algebra*. PhD thesis, Dept. of Electrical Engineering, 2002. www.utdallas.edu/~cantrell/matzke.pdf.

9. Bettencourt, Luis M.A. "The Origins of Scaling in Cities". Science v340, p. 1438. 21 June 2013.

10. Borthwick, D. and Garibaldi, S. "Did a 1-Dimensional Magnet Detect a 248-dimensional Lie Algebra?". arXiv:1012.5407v3 [math-ph] 25 Mar 2011

11. http://physicsworld.com/cws/article/news/2012/apr/19/first-hint-of-majorana-fermions-spotted-in-nanowires

12. https://www.quantamagazine.org/20150310-strange-stars-pulse-to-the-golden-mean/

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Appendix I [17]

Majorana Fermions in $\mathscr{G}_5$ over $\mathbb{Z}_3$

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Majorana fermions are characterized by having spin 1/2 and being self-conjugate, meaning that they are nilpotent, and thus are field-like forms of otherwise material particles.

Via exhaustive search of the complete set of tauquinions τ in $\mathscr{G}_5$, we find that all four fermions - protons, neutrons, electrons and neutrinos - have at least one Majorana operator M that transforms them into a self-conjugate form. That is, M is a discrete *form* whose sign-variants produce corresponding variants in the fermions.

$M = a + bcde$ is one such form in the case of an electron E:

$$E = -1 + ab + ac = -(a+b+c)a$$

where $(ab+ac)^2 = 1$ and $a+b+c$ is a photon. Then

$$ME = (-a+b+c) + abde - acde - bcde$$

$$EM = -(a+b+c) - abde + acde - bcde$$

and $(EM)^2 = 0 = (ME)^2$.

EM (for example) can be rewritten as

$$EM = -(a+b+c) - (ab+bc+ca)de$$

Note that $ab+bc+ca$ is a quaternion triplet, and because $abc(ab+bc+ca) = -(a+b+c) = (ab+bc+ca)abc$ and $-abc*abc = 1$, we can write

$$= -(a+b+c) - (ab+bc+ca)(-abc*abc)de$$

$$= -(a+b+c) - (a+b+c)abcde$$

$$= (a+b+c)(-1-abcde)$$

Recall now, from the main text, that m-vectors' squares cycle as powers of $i = \sqrt{-1}$ as m increases, ie. $mod\,4$. In this way, 1-vectors like $a+b+c$ and 5-vectors like $abcde$ have the similar algebraic properties, whence one can think of them as "octaves" of each other. [18] In the particular case of $(a+b+c)(-1-abcde)$ just above, we see that the Majorana electron's photonic aspect consists of a "fundamental" $-(a+b+c)abcde$ and a "*mod4*-octave" overtone $-(a+b+c)$. [19]

Or we can factor a out and in, and see the two concurrent electron phases,

$$= (-1+ab+ac)a + (-1+ab+ac)bcde$$

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[17] This appendix is in intended to be read in connection with the paper to which it is attached, ie. it is *not* self-contained.

[18] "...This description applies to the ground-state rotational band of carbon-12, but it also has significance for the Hoyle state. This is because *the spectrum of the Hoyle-state rotational band appears to be similar to that of the ground-state band* – with two of the five spin states measured already. However, the Hoyle state appears to have a *larger moment of inertia* than the ground state." [My italics] From http://physicsworld.com/cws/article/news/2014/jul/08/carbon-nucleus-seen-spinning-in-triangular-state.

[19] Or, if you prefer, a fundamental $a+b+c$ and a "*mod4*-octave" undertone $-(a+b+c)abcde$.

$$= EM = (-1 + ab + ac)(1 + abcde)a$$

whereas the latter equation tells us that a Majorana electron is an *a*-rotation/projection of that electron's combined fundamental and *mod4*-octave frequencies.

The $\mathscr{G}_5$ Majorana operator M has two basic forms, $\Sigma(v + wxyz)$ = *1+4* and $\Sigma(vw + xyz)$ = *2+3*, where the latter is a tauquinion field element τ, and the former one of the minus 1's in the tauquinion group. In particular, this minus 1 form is the linchpin connector between the magnetic and electric fields, ensuring that any true change in one results in a corresponding change in the other.

For example, in the case of $M = e - abcd$ and a field element, say $\tau = ab + cde$, we find that $M\tau = -cd - abe = \tau M$, so M inverts the signs of both the magnetic (2-vector) and electric (3-vector) components, *and* it interchanges them ($ab \leftrightarrow cd$), thus linking the two fields utterly. [Recall that the forms $vw + xyz$ are all quaternion isomorphs.]

In the case of the electron $E = -1 + ab + ac$, the *1+4* Majorana operators are:

$$M = a + bcde \qquad EM = (a + b + c)(-1 - abcde)$$
$$M = (b + acde) + (c - abde) \quad EM = (a + b + c)(-1 + abcde)$$
$$M = (b + acde) + (-c + abde) + (d + abce)$$
$$EM = (b - c + d)(-1 + e + abcd - abcde)$$
$$M = (a + bcde) + (b + acde) + (-c + abde) + (e - abcd)$$ [20]

The electron's *2+3* Majorana operators are:

$$M = \quad ad + bce$$
$$M = (\quad bd - acd) + (cd - abe)$$
$$M = (-ab + cde) + (ac + bde) + (-ae + bcd)$$
$$M = (-ab - cde) + (ac - bde) + (\quad ae + bcd) + (bc - ade)$$

So we see that the electron's Majorana operators are linear combinations of tauquernion group elements. Similarly for the proton $P = -1 + a + b + c + ab + ac$, the corresponding Majorana operator is three *2+3*'s:

$$M = (-ae + bcd) + (-bd + ace) + (-be - acd) \qquad PM = -ae - be - ce + abd - acd + bcd$$

and likewise for the neutron $N = abcP = b - c + ab - ac + bc - abc$,

$$M = (-ae + bcd) + (-bd + ace) + (-be - acd) \qquad NM = -ad - bd - cd - abe + ace - bce$$

whose M is identical to the proton's.

In the case of the neutrino $n = -1 + a + b + ab$, [21] the corresponding Majorana operators are:

$$M = (\quad a - bcde) + (-b - acde)$$
$$M = (-c + abde) + (-d - abce) + (-e + abcd)$$
$$M = (\quad ad + bce) + (\quad bd - ace)$$
$$M = (\quad cd - abe) + (\quad ce + abd) + (de - abc)$$

[20] The rightmost factors, $(-1 \pm abcde)$ and $(-1 + e + abcd - abcde)$, are idempotents and a square root of same (*sqert* [7]), respectively.

[21] I note that Appendix I in [7] supports the existence of a fourth neutrino as a linear combination of three basic phases, and strictly combinatorially there could be three more phases via pairwise combination.

$$M = (-ac - bde) + (-ae + bcd) + (cd + abe) + (-de + abc)$$

For the first of these, $nM = a + ab + cde - bcde = (1-b)(a+cde)$, where the rightmost *product* is the *process* that generates the *state* to its left.

Thus there are Majorana operators in $\mathscr{G}_5$ for all four fermions, and all of these operators are linear combinations of either the *1+4* or the *2+3* elements of the tauquinion group.

This in turn prompts the question, Are there Majorana fermions in the tau*quernion* (ie. gravitational) field too? Yes. And indeed, every category has at least one valid Majorana example except *2+2* for protons and neutrons. We list a few examples:

$\mathscr{G}_4$: *Tauquernion Majorana Operators*

1+3

Electron	$d - abc$
	$-b + c - abd - acd$
	$-a + b + d - abc - acd + bcd$
	$-a + b - c + d - abc - abd - acd - bcd$
Neutrino	$-c - d - abc - abd$
	$a - b + c + abd + acd + bcd$
Proton	$-b + c - d + abc + abd + acd$
Neutron	$-b + c + d - abc + abd + acd$

2+2

Electron	$ad + bc$
Neutrino	$(-ac - bd) + (-ad - bc)$
Proton	*None*
Neutron	*None*

It is also intriguing that some combinations generate Higgs bosons [7]:

For $E = -1 + ab + ac$ and $M = ad + bc$,

$$EM = -ab + ac - ad - bc - bd - cd = H$$

And for $N = b - c + ab - ac + bc - abc$ and $M = a + b + c - abd + acd - bcd$,

$$NM = ab - ac + ad + bc + bd + cd = H$$

Note also that these are complementary, ie. they sum to zero.

In general, while there are a few thousand of these discrete tauquinion (*1+4* and *2+3*) sign-variants of M altogether, the tauquernion variants (*1+3* and *2+2*) number in the hundreds; details are available on request. We remind the reader that [7] identifies the *1+3* forms $w + xyz$ as dark matter.

One can hope that the present computationally distributed and *combinatorially exact* description of quantum mechanics, which is consistent with the Standard Model, will be useful in topological quantum computation.

Because our $\mathbb{Z}_3$ geometric algebra - being both finite and discrete - is a *literal* representation of *actual* events (ie. state changes), each of the various possible factorizations represents a different pattern of actions leading to the Majorana particle, eg. EM. It should be clear that in this algebra's finite and discrete computational space, there is no room to hide from exhaustive search, and there is nothing left out, eg. $\mathscr{G}_3$ is isomorphic to the Pauli algebra , etc. And as pointed out earlier, products express *actual processes*, just as sums express their concurrency.

One can think of the algebra, interpreted and used in this way, as a very sophisticated programming language, one that *specifies exactly what actually happens* while automatically maintaining the semantics of non-deterministic outcomes. We believe this algebra to be superior to the braid algebras usually applied in this context because the latter, despite their formal abstraction, have a strong sequential feel nevertheless, which is unhelpful. Appositely, $\mathscr{G}$ specifies actual, truly concurrent mechanism.

Thus these factorizations are not just mere formal manipulations, but rather, due to the algebra's literality, they are *different structural views* of the same *computational* object, just as one gets different views of a traffic intersection from different vantage points (eg. driver vs. green-arrow scheduler). Since the algebra is both discrete and finite, the statistics of these patterns and processes should match those of actual experiment.