Riemann Fever

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Abstract. The confluence of computer science and quantum mechanics has inspired a proof of the Riemann Hypothesis for the discrete geometric (Clifford) algebras $Cl(n,0)$ over $\mathbb{Z}_3 = \{0,1,-1\}$, as expected from the Weil Conjectures. The symptoms of Riemann Fever are clearly evident here - long periods of apathy punctuated by febrile investigations lasting from minutes to (here, two) weeks, accompanied by alternating explosions of euphoric insight into cosmic truths, and insight becalmed in a sea of profundity; and as well a weakness for sweeping conclusions of great import. I recount the course of the affliction in my own case of this recurrent virulent ailment, which often leads to addiction to further attacks, and other complications.

Foreword

My first contact with the Riemann Fever virus (though I of course didn't realize it at the time) was in high school via Dantzig's book *Number - the Language of Science* [11]. It was here that I first (I think) encountered the concept of the prime numbers and the mysteries surrounding them. The exposure was brief, but the virus nevertheless took hold. It was strengthened intermittently over the next ten or so years, which interval ended with me a computer scientist, software division.

There things stood for some 20 years, whence my interest in concurrent systems eventually led me to *geometric* algebras (aka. Clifford algebras). Quite unbidden, a very early thought here was, and I quote the sober little voice in my head, *There's something about the primes here*. Ach!! This was the early 1990's. But once again the outbreak was minor, as also with a 1996 item in Science magazine [1] about a connection between the primes and particle physics, recounted below, that one might otherwise expect to have resulted in a major attack.

Another ten or so years passed, and finally the toll extracted by the many lesser encounters over the years culminated in my acquiring a copy of John Derbyshire's excellent book *Prime Obsession* [2]. Hereafter, an incubation period of a couple of years resulted in the full-blown incident of Riemann Fever that is recounted in the following. It all began innocently enough, so read further at your own risk... [1]

[1] I have done only minor clean-up and organization of the record, plus added references.

Introduction

In 1996 Science magazine reported [1] a connection between analytic number theory and quantum physics whose core is the Riemann zeta function $\zeta(s)$, a statement intimately connected to the distribution of the primes. This result attracts attention because it is quite unexpected - or perhaps entirely *right* - that a purely number-theoretic property, primality, should have anything to do with the basic structure of reality. However, the probablistic nature of both the prime number distribution and the quantum formalism itself unfortunately enshrouds the exact nature of the connection.

To this situation we bring the *synchronizational* model of quantum mechanics [5,6], which by being directly computational, provides a concrete and detailed *mechanism* for quantum processes. By "directly computational" is meant that geometric algebra functions *(1)* as a program specification language, *and (2)* as a language that allows the symbolic manipulation of this same program "code" according to the usual mathematical operations, *and (3)* as a language describing physical reality, all simultaneously. The underlying computational power of the model (being quantum mechanical) exceeds that of classical Turing machines, and constitutes a considerable extension of semantic power. The statistical aspect of contemporary physical theory is, in the present view, the result of the combinatorics of the various discrete algebraic possibilities.

Understand as well that the model describes and applies to *any* distributed system, including, especially, self-organizing ones.

More precisely, we use the discrete and finite geometric algebras $Cl(n,0)$ over $\mathbb{Z}_3 = \{0,1,-1\}$, where the ± 1 captures presence/absence, spin, charge, etc. at the price of compacting relationships based on multiplicity; zero represents *Void*; and synchronization is encoded by idem- and nil-potent elements. Geomeric algebra, being inherently hierarchical and well-suited to expressing and tracking symmetries and their transformations, has turned out to be a very apt vehicle for the description of both physics and computation; see [7,8,9; 5,6].

Notation. Let x_i be a unit 1-vector, where $|x_i| = (x_i)^2 = +1$; and specify that $x_i x_j = -x_j x_i$ for $i \neq j$. Then the set $X = \{x_1, x_2, ..., x_n\}$ generates the geometric algebra $Cl(n,0)$. Both addition and multiplication are associative, and multiplication distributes over addition as usual; the result of a multiplication can be understood as a rotation of the one factor by the other. Note that the Lie bracket $[A,B] = AB - BA$ is naturally defined in geometric algebras. For readability we occasionally write $a,b,c,...$ for the x_i. Since multiplication is in general not commutative, our convention is "operate on the left on the operand to the right".

The hierarchical aspect of the geometric algebras is realized via the graded dimensionality of its mutually orthogonal primitive elements 1, $\{x_i\}$, $\{x_i x_j\}$, $\{x_i x_j x_k\}$, $..., x_1 x_2 ... x_n$, and together these form a basis for the $\Sigma\binom{n}{m} = 2^n$- dimensional combinatorial space of *same-different* distinctions that we will be working in. Sums in the algebra represent the simultaneous existence of said elements (eg. *states*), and a product AB is understood to be element A operating on element B. In [5,6] we use the co-boundary operator δ to construct higher-grade elements from lower grades[2] and so the algebra's graded

[2]Via the criterion *if* $|\partial_Y X| = |YX| = |Y'|$ *then* $X = \delta Y$ *and* $\partial_{Y'} X = Y$. So, for example, the quark-form

hierarchy

$$\{1, a, b, c, d..., ab, ac, ad, ..., bc, bd, ..., abc, ..., abcd, ...\}$$

should be seen in this constructive, structural light.

Calculating Frequencies

Since the elements of the algebra represent, in general, concrete physical entities, it is reasonable to require of them that they can be treated as having a wave-like aspect. For example, what is the frequency spectrum and oscillatory behavior of an arbitrary element of the algebra, eg. $a+b+ab$? To answer this, view the fundamental *boundary* of the hierarchy as the primitive sensor vector $x_1+x_2+x_3+...$, and note that each x_i must *necessarily* (because finite) oscillate between the *discrete* values $+1$ and -1.[3]

Such an x's oscillatory behavior can be captured by the discrete scalar function $sin(\phi+\omega t)$, where ω is the angular velocity, t is a monotonically increasing time counter, and ϕ a phase displacement (choose $\phi=0$). Thus ωt is a length, and taking $t=1$ lets ω carry the concept of wavelength (and thus frequency) without running into temporal and measurement issues.

We wish to calculate $\sum sin(x_i)$ - the oscillatory behavior itself - and $\sum f_i$, the system's frequency spectrum. Recall that frequency f and wavelength λ are inversely related: $\lambda = \frac{1}{f}$.

One might think that $\sum sin(x_i)$ can be calculated via the identity

$$sin(A)+sin(B) = 2\,sin(\tfrac{A+B}{2})\,cos(\tfrac{A-B}{2})$$

Unfortunately, it's unobvious how extend this to $sin(A)+sin(B)+sin(C)+...$, although *if* it can be done, associativity guarantees that we will ultimately get to the same (top) node, the various paths - associatively speaking - are the same as the co-boundary relations and will yield the specific behaviours of each corresponding node.

Calculating Σf_i, or rather $\Sigma \lambda_i$, is more fruitful:

$$\sum \lambda_i = \lambda_1+\lambda_2+... = \tfrac{1}{f_1}+\tfrac{1}{f_2}+...$$

Frequencies add as

$Y = a+bc$ induces the existence of the charge carrier $X = abc$, since $YX = (a+bc)(abc) = -a+bc$ and $|a+bc| = |-a+bc|$.

[3] That is, think of the hierarchy as being a reactive computational entity embedded in a surround that it senses via these one bit sensors: $x=+1$ denotes that whatever x senses is currently present in the surround, and $x=-1$ means conversely that whatever x senses is currently *not* present in the surround. The x_i are then combined via δ, recursively, to generate the hierarchy. Zero means "the computation ends", or "does not occur" ie. being *of* neither time-like nor space-like character.

$$\frac{1}{f_1}+\frac{1}{f_2}=\frac{f_1+f_2}{f_1 f_2} \quad \text{and} \quad \frac{1}{f_1}+\frac{1}{f_2}+\frac{1}{f_3}=\frac{f_1 f_2+f_1 f_3+f_2 f_3}{f_1 f_2 f_3}$$

and in general, letting $F=\{f_1,f_2,...,f_n\}$ and abusing combinatorial notation, write

$$\sum \lambda_i=\lambda_1+\lambda_2+...+\lambda_n=\frac{1}{f_1}+\frac{1}{f_2}+...+\frac{1}{f_n} \;=\; \frac{\Sigma\begin{pmatrix}F\\ n-1\end{pmatrix}}{\begin{pmatrix}F\\ n\end{pmatrix}}$$

The frequencies f are a *scalar* property of the 1-vector sensors that form the boundary of a system. The simplest case is when the f_i all have the same frequency, denoted by "1" in the following table of hierarchically constructed frequencies:

Level Transition δ	$\lambda=\omega_1+\omega_2 \;=\; \frac{1}{f_1}+\frac{1}{f_2} \;=\; \{f_1,f_2\}$	λ	f	*Interval*
a	1	1	1	root
$a+b\to ab$	$\{1,1\}$	2	.5	octave
$a+bc\to abc$	$\{1,\{1,1\}\}$	$\frac{3}{2}$	.66	fifth
$a+bcd\to abcd$	$\{1,\{1,\{1,1\}\}\}$	$\frac{5}{3}$	.6	major sixth
$a+bcde\to abcde$	$\{1,\{1,\{1,\{1,1\}\}\}\}$	$\frac{8}{5}$	.625	minor sixth
$ab+cde\to abcde$	$\{\{1,1\},\{\{1,\{1,1\}\}\}$	$\frac{7}{6}$	.86	minor third
$ab+cd\to abcd$	$\{\{1,1\},\{1,1\}\}$	1	1	octave as root
$abc+def\to abcdef$	$\{\{1,\{1,1\}\},\{1,\{1,1\}\}\}$	$\frac{4}{3}$	.75	fourth
$abcd+efgh\to abcdefgh$	$\{\{\{1,1\},\{1,1\}\},\{\{1,1\},\{1,1\}\}\}$	2	.5	octave's octave

Note that we are creating *under*tones, not *over*tones, so the wavelengths *grow* by the ratios shown, and the frequencies *fall* similarly. This reflects the fact that the higher the level of the hierarchy, the more global and "longer view" it reflects of the system's interaction with its surround.[4]

The physical connection & Riemann

The energy difference $E_\Delta=hf_\Delta$ between the first and second frequency levels in the above table is $1-0.5=\frac{1}{2}$. Similarly, the second-to-third frequency level increment can be any one of $.66-0.5=\frac{1}{6}$, $2\times\frac{1}{6}=\frac{1}{3}$, or $3\times\frac{1}{6}=\frac{1}{2}$, depending on how many of the co-occurrences $a+bc$, $b+ac$, $c+ab$ are present. The f_Δ from level 1 to level 3 is $1-.66=\frac{1}{3}$, and thence $\frac{2}{3}$ and $\frac{3}{3}=1$. The $\mathbb{Z}_3$ algebra normalizes Planck's constant h to 1, but decimally, $E_\Delta=\frac{1}{2}h$. Given the hierarchical context, this energy would then constitute the inherent rest-energy (a stand-in for mass at this stage of the hierarchical construction) of particles.

[4]It's interesting that the pure-ratio musical intervals show up, although the occurrence here of small-number ratios in general is to be expected. Still missing: major 2nd ($\frac{9}{8}$ or $\frac{10}{9}$, $\frac{8}{7}$), minor 2nd ($\frac{16}{15}$, $\frac{25}{24}$), major 3rd ($\frac{5}{4}$), minor 7th ($\frac{16}{9}$, $\frac{9}{5}$, $\frac{7}{4}$), and major 7th ($\frac{15}{8}$, $\frac{48}{25}$). Cf. Harry Partch, *Genesis of a Music*, p. 68.

Howsoever, since Planck's constant h introduces a fundamental physical discreteness, we can reason that the shortest wavelength is the Planck length $\tilde{h}$, so $\lambda_1 = \tilde{h}$, and thus that an arbitrary wavelength $\lambda_m = m\tilde{h}$, whence $f_1 = \frac{1}{\tilde{h}}$ and an arbitrary frequency $f_m = \frac{1}{m\tilde{h}}$. Substituting this into the sum, we find that the overall spectrum is

$$\sum \lambda_i = \lambda_1 + \lambda_2 + ... = \frac{1}{f_1} + \frac{1}{f_2} + ... = \frac{1}{\tilde{h}} \sum_{m=1} \frac{1}{m}$$

It is, in fact, the (divergent) *harmonic series*: $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + ...$. In the interests of generality [cf. Riemann], rewrite the sum as

$$\zeta(s) = \sum_{m=1} \frac{1}{m^s} = 1 + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \frac{1}{5^s} + ...$$

where s is complex and the left-most Greek letter is *zeta*; the series converges for all $s > 1$. Following Euler's classic manipulation [2], multiply both sides of the above equation by $\frac{1}{2^s}$ and subtract the result from the original to get

$$(1 - \frac{1}{2^s})\zeta(s) = 1 + \sum \frac{1}{m^s} = 1 + \frac{1}{3^s} + \frac{1}{5^s} + \frac{1}{7^s} + \frac{1}{9^s} ...$$

Note that *all* terms containing a multiple of two have disappeared from the rhs. Now iterate this process infinitely with $\frac{1}{3^s}, \frac{1}{5^s}, ...$, and divide to get

$$\zeta(s) = \prod_p (1 - \frac{1}{p^s})^{-1} = (\frac{1}{1-\frac{1}{2^s}})(\frac{1}{1-\frac{1}{3^s}})(\frac{1}{1-\frac{1}{5^s}})(\frac{1}{1-\frac{1}{7^s}})(\frac{1}{1-\frac{1}{11^s}})(\frac{1}{1-\frac{1}{13^s}})...$$

where p ranges over the primes. It now follows that

$$\sum_m \frac{1}{m^s} = \zeta(s) = \prod_p (1 - \frac{1}{p^s})^{-1}$$

Thus an infinite sum over the integers equals an infinite product over the primes! It is therefore not surprising that this *Riemann zeta function* $\zeta(s)$ is intimately related to fundamental issues in prime number theory. Interpreting this physically, summing $\zeta(s)$ [in its Σ-form] over all s is some sort of universal wave function.

The last variation on the theme is the Möbius function, μ, the reciprocal of ζ:

$$\mu(s) = \frac{1}{\zeta(s)} = \sum_{m=1} \frac{\mu(m)}{m^s}$$

$$= ...(1 - \frac{1}{11^s})(1 - \frac{1}{7^s})(1 - \frac{1}{5^s})(1 - \frac{1}{3^s})(1 - \frac{1}{2^s})$$

$$\mu(s) = 1 - \frac{1}{2^s} - \frac{1}{3^s} - \frac{1}{5^s} + \frac{1}{6^s} - \frac{1}{7^s} + \frac{1}{10^s} - \frac{1}{11^s} - \frac{1}{13^s} + \frac{1}{14^s} + \frac{1}{15^s} + ...$$

The terms of μ follow the sign-rules:

$$\mu(m): \begin{array}{ll} \mu(1) & = 1 \\ \mu(m) & = 0 \text{ if } m \text{ has a square factor (which eliminates all } m^s,\ s>1) \\ & \\ \mu(m) & = \text{-1 if } m \text{ is a prime or the product of an } \textit{odd} \text{ number of different primes} \\ \mu(m) & = \text{+1 if } m \text{ is the product of an } \textit{even} \text{ number of different primes} \end{array}$$

Recalling the path we have followed, we began in the multi-vector world, asking the question, What is the frequency spectrum of (say) $a+b+ab$? This led us to the harmonic series, which led us to ζ and μ, both of which are, however, creatures of the *scalar* world. Thus, dimensionally speaking, we have *descended* from 2^n- dimensionality to the single dimension of the scalars = $Cl(0,0)$.

From physics to computation

So the question arises, is it possible to translate these scalar relationships *back* into vector language? *Yes* - by re-doing Euler's derivation, but this time in $Cl(n,0)$ over $Z_3 = \{0,1,-1\}$. Anticipating what is to come, we can drop the complex s because we have the imaginaries automatically with the geometric algebra, since for $i \neq j$, $(x_i x_j)^2 = -1$; furthermore, we can drop s itself because the idempotent property we will employ flattens all exponents to 1. Nevertheless, the key *factor* properties remain, and it will become clear that it is idempotents that correspond to the scalar primes.

We begin with an initial "state vector" $S_0 = 1 + q_2 + q_3 + ...$ and require that $q^2 = q$, ie. that q be idempotent. Now manipulate, just as before, to get $S_1 = S_0 - q_2 S_0 = (1-q_2)S_0$. In general,

$$S_i = S_{i-1} - q_{i+1}S_{i-1} = (1-q_{i+1})...(1-q_3)(1-q_2)S_0 = PS_0$$

where P is the product $(1-q_{i+1})...(1-q_3)(1-q_2)$. The q's *must* be idempotent because if (eg.) $q^2 = \pm 1$, then the fact that q is a factor of P is quickly lost, and with it the subtractive cancellation of q's multiples. The sum S_0 *cannot* have finite length because each multiplication generates terms that only match terms further out, so S_0 must always be longer than $q_{i+1}S_{i-1}$ if the desired subtractive cancellations are to occur.[5]

Suppose, now, that $q_i = -1 + u_i$, where $u_i^2 = 1$, and further, associate the integer i with the vector q_i so that $q_{jk} = q_j q_k$, for example $q_6 = q_2 q_3$. Then

$$\begin{array}{c} S_0 = \\ 1+(-1+u_2)+(-1+u_3)+(-1+u_2)^2+(-1+u_5)+(-1+u_2)(-1+u_3)+(-1+u_7) \\ + \quad (-1+u_2)^3+(-1+u_3)^2+(-1+u_2)(-1+u_5)+... \end{array}$$

[5] Although there may be a practical work-around, eg. require $q^r = 1$, whence r puts a ceiling on meaningful i. The situation reminds one of the "renormalization" issue in quantum field theory.

corresponds to the sequence $1,2,3,4,...$. Now multiply through by "2" = $q_2 = (-1 + u_2)$ to get

$$(-1+u_2)S_0 = (-1+u_2) + (-1+u_2)^2 + (-1+u_2)(-1+u_3) + (-1+u_2)^3 +$$
$$+ \quad (-1+u_2)(-1+u_5) + (-1+u_2)^2(-1+u_3) + (-1+u_2)(-1+u_7) + ...$$

and subtract this from S_0 to get

$$S_1 = (-1-u_2)S_0 = 1 + (-1+u_3) + (-1+u_5) + (-1+u_7) + ... = \bar{q}_2 S_0$$

where $\bar{q} = -1-u$ is the conjugate of $q = -1+u$. Clearly, all the "even" terms - those with the factor $(-1+u_2)$ - have disappeared from the rhs and only the "odd"-numbered terms remain. Now repeat the process with q_3 to get

$$S_2 = (-1-u_3)(-1-u_2)S_0 = 1 + (-1+u_5) + (-1+u_7) + ... = \bar{q}_3\bar{q}_2 S_0$$

which rhs is missing all terms having "3" = q_3 as a factor. Clearly, if we continue this process, we will arrive at

$$S_k = P_{k+1}S_0 = \bar{q}_{k+1}...\bar{q}_3\bar{q}_2 S_0 = 1 + (-1+q_{k+2}) + (-1+q_{k+3}) + ...$$

At this point, Euler argues [in the scalar case] that the remaining terms on the rhs all eventually disappear, and hence that one could [analogously] write $P_{k+1}S_0 = 1$; but this is not possible here because $P_{k+1} = \bar{q}_{k+1}...\bar{q}_3\bar{q}_2$ is not a reversible process and hence *cannot* have S_0 as an inverse.[6,7] Rather, one must argue that the frequency of the remaining terms (after some point) is so low as to be insignificant.

Even with this, however, we *still* cannot divide through by P_{k+1} to get $\zeta = \frac{1}{P_{k+1}}$, ie. the zeta function for $Cl(n,0)$, because ignoring small terms doesn't change the fact that the very concept P_{k+1}^{-1} is meaningless when P_{k+1} by definition *has* no inverse. Rather, we must be satisfied with ζ's inverse, namely $\mu = P_{k+1}S_0$. Despite this, however, the zeroes of μ are in the same places (ie. arise from the same factors $\bar{q}$) as the zeroes of ζ, since it's the same P; we will return to this later.

At this point, rather, the question is, what does P_{k+1} look like when multiplied out? This polynomial can be constructed combinatorially: define the set $U = \{\bar{u}_2, \bar{u}_3, \bar{u}_5, ...\}$, let m be the grade of the m-vector, and as before, abuse combinatorial notation to write

$$\mu = 1 + \sum \mu_j = 1 + \sum_{n=1} (-1)^m \begin{pmatrix} U \\ n \end{pmatrix}$$

[6]Theorem: For all $X \in Cl$, X has no inverse *iff* X has an idempotent factor.

[7]Notice, by the way, the fascinating interplay between the concepts of constructiveness and of a limit at ∞ versus physical ir/reversibility - only in the (by definition *unrealizable*) limit is the 2^{nd} Law of Thermodynamics ultimately violated.

$$= 1 - (u_2 + u_3 + ...) + (u_2u_3 + u_2u_5 + ...) - (u_2u_3u_5 + u_2u_3u_7 + ...) \pm ...$$ [8]

Note the following properties of this sum:

1. If there are an odd number of u-factors, then the sign of the m-vector term will be negative because each u has negative sign.

2. If there are an even number of u-factors, then the sign of the m-vector term will be positive.

3. There are no multiples of any q_i present.

These are precisely the conditions that define the scalar version of μ.

It should be noted that the conjugate forms that are an implicit part of this story follow naturally from the different possible orderings in the "non-prime" $q-$terms in S_0. Alternatively, the order in which the q_i are accumulated into P will also generate the conjugates: one could have either $(-1+u)(-1+v) = 1-u-v+uv$ or $(-1+v)(-1+u) = 1-u-v-uv$.

Thus $(-1 \pm u)(-1 \pm v)(-1 \pm w)$ has actually six different orderings, and all of these are members of the same (irreversible) conjugate set of order $2^n n!$. [9]

The Riemann Hypothesis for $Cl(n,0)$

This form of the μ-function (ie. with $q_i = -1 + u_i$, where $u_i^2 = 1$) produces a polynomial each of whose terms is a unique singleton m-vector $u_i...u_j$, each of which has magnitude $|u_i...u_j| = 1$. Furthermore, all these m-vectors are mutually perpendicular, and so a sum of such terms is the diagonal of the corresponding hypercube and therefore must satisfy the Pythagorean relation

$$|\mu_1| + |\mu_2| + ... + |\mu_k| = \left[|\mu_1|^2 + |\mu_2|^2 + ... + |\mu_k|^2 \right]^{\frac{1}{2}}$$

Let ℓ count the number of q-factors currently making up μ in its product form. Then, at any such stage ℓ of μ's growth, sum the 1's of $\mu^{(\ell)}$'s sum-form terms:

$$M(k) \;=\; \sum_{j=1}^{k} |\mu_j| \;\leq\; \left[\sum_{j=1}^{k} |\mu_j|^2 \right]^{\frac{1}{2}} \;=\; \mathscr{O}(k^{\frac{1}{2}})$$

[8] Lest the reader worry that this Clifford μ doesn't show fractions as does the original scalar μ, notice that the m-vector terms are all self-inverse, squaring namely to ± 1, whence $\mu_i^{-1} = \pm \mu_i$. So this equation could as well be written as $1 - \frac{1}{u_2} - \frac{1}{u_3}...$ (except, of course, that geometric algebras don't do "division"). The inverse relationship between wavelength and frequency is thus maintained quite literally.

[9] Which for $n = 3$ yields 48; $Cl(3,0)$ is isomorphic to the Pauli algebra.

where k is the number of terms. $M(k)$ is called Merton's function [2], and by showing that $M(k) = \mathscr{O}(k^{\frac{1}{2}})$, we have proven the Riemann Hypothesis for the spaces covered by $Cl(n,0)$. [10] This is the expected result, due to DeLigne's proof of the Weil Conjectures, which cover a wide range of spaces [3,4]; see the Appendix.

It is worth pointing out that $\bar{q} = -1-u$ is written in $\mathbb{Z}_3$, whereas in the real numbers $\mathbb{R}$, the $\mathbb{Z}_3$ minus one becomes $-1 = \frac{1}{-1} \mapsto \frac{1}{2}$, that is, all the non-trivial roots of μ, and therefore of ζ, lie on the line $x = \frac{1}{2}$. Presumably, the mapping $-1 \mapsto 2$ speaks to the trivial "even" roots of ζ. These two mappings match the usual statement of the Riemann Hypothesis: that all the non-trivial roots of ζ lie on on the real line $x = \frac{1}{2}$.

Applications

We have thus arrived at the relationships

$$\mu = P_{k+1}S_0 = (\bar{q}_{k+1}...\bar{q}_5\bar{q}_3\bar{q}_2)(1+q_2+q_3+q_4...) \quad = \quad 1+q_{k+2}+q_{k+3}+...$$

$$=$$

$$(1-(u_2+u_3+...)+(u_2u_3+u_2u_5+...)-(u_2u_3u_5+u_2u_3u_7+...)\pm...\pm u_2u_3...u_{k+1})S_0$$

In the following, we try to tease some useful insights out of these. We will henceforth refer to P_{k+1} generically as P.

[It is well at this point to recall that the components $q_i = -1+u_i$ of the state S_0 possess implicitly *both* a discrete particulate aspect - in that the u_i are unitary - *and* a wave-like aspect - being finite, the u_i's magnitudes can only oscillate.]

The Process Concept

Thinking back to how P was created, the original sum $S_0 = \sum q_i$ with $q^2 = q$, and particularly $q_{jk} = q_j q_k$, in effect specifies all possible products of the individual idempotent q's. Since these products are not commutative, there is an ambiguity as to their ordering; we allow any ordering. There is a second ambiguity in the order in which the various q's are chosen to become the factors of P, and again, we allow any ordering. As noted earlier, these various orderings are Cl's way of generating conjugate forms. Howsoever, with all possible orderings and the fact that the q's are irreversible, we have *all possible time-like processes*.

Looking at this computationally, and recalling that the mathematical expression ***is*** the literal mechanism of the computation, the rhs $1+q_{k+2}+q_{k+3}+...$ is *the state arrived at* after the application of the process/operator P to an initial state S_0. Indeed, aside from the fact that S_0 is not unitary, the form $PS_0 = 1+q_{k+2}+q_{k+3}+...$ describes a measurement P made on a state S_0 that yields the result "Yes *[= the +1]* , the state S_0 was found in the environment, which by the way is now equal to $q_{k+2}+q_{k+3}+...$".

[10] That is, for $n > 0$. If $n = 0$, we are in the scalars, whose only idempotents are $0^2 = 0$ and $1^2 = 1$, whence the $q_i = +1$ and $S_0 = 1+\Sigma 1$, which in $\mathbb{Z}_3$ is ill-defined, since the sum can be any of $0, 1$, or -1.

A geometric algebra's multiplicative associativity masks the necessarily absolute sequentiality of the physical process, cf. the group E_8, which is namely *not* associative. The lack of associativity can namely enforce (eg.) a sequential right-to-left application of P's factors q_i to the current state. The price of directly tracking this aspect of physicality is, however, the loss of both manipulative and conceptual flexibility, the latter because it welds the use (and the physics) into a sequential view of what is in truth a distributed concurrent system that *cannot*, even in principle, be so captured (see the *Coin Demonstration* in [6]).

Put differently, by willy-nilly enforcement of sequentiality via non-associativity, one loses the natural (and very useful) *non*-specification of order-of-evaluation of a geometric algebra. Of course, one can explicitly specify the useful exceptions in either case, but at least in the computational case - where one is namely mostly interested global properties over all possible orderings - associativity is the more useful.

Synchronization & Causality

We can say that P is a *sequential computational process* because it is established in [5] that each of P's component actions, the idempotent operators q_i, is semantically equivalent to the computational *synchronization* operator ***Signal***(u_i). The *Signal* operation is paired with ***Wait***(u_i), which is *nilpotent*, and the pair correspond to the fermion/boson distinction in particle physics. Together they constitute the necessary and sufficient operators to construct any time-like, irreversible, computation - including "parallel" and "distributed" ditto, and as well the construction of memory and *if-then-else* [5].

Being nilpotent, ***Wait*** has no inverse. So, via the idempotence-*iff*-no-inverse theorem footnoted earlier, it follows that every nilpotent is grounded in a matching idempotent. For example, the nilpotent $-a+b+c$, a photon, is related to the electron $ab+ac$ via $a(-a+b+c)$. One does the following to get a nilpotent out of an idempotent.

The key is two unitaries that anti-commute: if U,V are two such unitaries, then all sign-variants of $U+UV$ are nilpotent. Since $U^2 = V^2 = 1$ and $UV = -VU$, the operate-on-the-left sequence

$$(-1+V)(-1+U)$$

can, using the identities

$$(-1+V) = (-1+V)(-V) \quad \text{and} \quad (-1+U)^2 = (-1+U)$$

be converted into $(-1+V)(-V)\,(-1+U)(-1+U)$

$$= (-1+V)\,(V-VU)\,(-1+U)$$

where $W = V - VU$ is the desired nilpotent, playing the role of ***Wait***(U). The corresponding computational process is ***Signal***(U); ***Wait***(U); ***Signal***(V), where we now read left-to-right in computer program order:

First, *signal* U ...

"whilst in some other process"

... *wait* for U to occur, and only then *signal* V.

Thus, for the event ***Signal***(V) to be causally connected to the (preceding) event ***Signal***(U), there must exist the mediating nilpotent entity W. Indeed, if such a W doesn't exist, then the two events $-1+U$ and $-1+V$ aren't causally connectible.[11,12]

So, to cover all possible causal sequences in the present case, we must have at our disposal all the "prime" idempotents $\overline{q}_i = -1 - u_i$ and all the u_j that anti-commute with each u_i. Then the sequence $(-1-u_i)(-u_i - u_i v_j)(-1-v_j)$ exists and is a causal sequence for all u_i, u_j.

Arbitrary Idempotents

Notice that *any* choice of the idempotents $q = (-1+u)$ will generate the Riemannian result.

Example. The 1-vector product $(-1+u)(-1+v)(-1+w)...$ generates a complete polynomial, ie. one having all 2^n possible m-vectors. This set of idempotents satisfies all the steps of the above proof, but so do the similar products of 4-vector idempotents $-1+wxyz$, of 5-vector idempotents $-1+vwxyz$, etc. So conceivably the appearance of the scalar prime number statistics in the physical context is the result of the structure of the argument itself, rather than an appearance of the scalar primes *per se* (although I do not myself believe this).

Howsoever, this reveals (or perhaps merely systematizes) a way to model, at a strictly higher level, things like atoms and molecules, both of which clearly exceed the expressive power of $Cl(3,0)$. In a software specification and development context, it specifies how to structure a distributed system and verify its functionality one level at a time.

Example. The unitary u_i's have been written in lower case to encourage the interpretation that the u_i are *singleton* m-vectors $x_i...x_j$, but this need not be the case, as this property was never invoked in the above. Rather, the u_i can also be multi-vectors like $U_i = u+v+uv$ or $U_j = uv+uw$, which are also unitary. This means that the products, like U_iU_j, in the sum-form of μ can generate multiple terms, but these new μ_i do not change the Pythagorean result and the conclusion that $\Sigma|\mu| \leq \mathcal{O}(k^{\frac{1}{2}})$.

In a software context, the coalescence of such *compound* unitaries out of the sea of singleton unitaries is an *implicit* and *unavoidable* implication of the μ-hierarchy, *and* they

[11] One can understand such connecting nilpotents in a very concrete computational way, namely that each corresponds to a trip around the hardware interpretive *Ifetch* loop, which causally connects two successive instructions (= the idempotents). Alternatively, one can think of an idempotent $-1+U$ as being the *self*-boundary of its associated unitary U: $\partial_{-1+U}U \cong -1+U$, which can only be connected to another such self-boundary $-1+V$ by an entity that namely *has* no self-boundary, namely a nilpotent, since $\partial_W W = WW = 0$.

[12] Let ";" mean causally connected and ":" mean *not* so connected; and let A,B,C,D be idempotent. Then the sequence $(A;B):(C;D)$ can be rewritten as $(AB)+(CD)$. Similarly, the sequence $A;(B:C);D$ could mean any of $A(B+C)D$, $A(BD+C)$, or $A(B+CD)$, depending. The algebraic notation is more precise and flexible.

themselves can be the subjects of causal sequences $...(-1+U_j)(-1+U_i)...$. This in turn prompts the question: what do we do about these?? Are they an opportunity to expand the object concept, and to build a meta-hierarchy, corresponding to the hierarchy of the elements, molecules, meta-materials, cells ... ?

Or, like undocumented logic, are these implicitly-appearing compound unitaries a threat to privacy and security? Note that as long as a given U_i is never actually *assigned* a name, it cannot be explicitly addressed, and so both appears and dissolves automatically. Verification of system functionality, ie. the *proof* that it does or does not do X, is simplest when compound unitaries are denied objective existence, since then the system has a very tractable wave-like mathematical structure. On a practical level, this reversibility greatly improves system stability. Once assigned a name, however, a compound unitary becomes reified, and requires that the system now *ensure* its persistence and accessibility; the chance of deadlock escalates, while at the same time the entity's very existence encourages the creation of ever more irreversible processes.

Universal Hierarchy

Notice that the terms of the sum-form of P express all $\left(\begin{array}{c} n \\ m \end{array}\right) = 2^n$ m-vectors, and hence is a basis of the space spanned by $Cl(n,0)$. Computationally, this space is the space of all possible distinctions.[13]

With this in mind, take the μ-form of our relationships and suppress the detail by defining various Q's and U_{P_k}:

$$\mu = P_k S_0 = (\bar{q}_k...\bar{q}_5\bar{q}_3\bar{q}_2)(1+q_2+q_3+q_4...) \quad = \quad 1+q_{k+1}+q_{k+2}+... = 1+Q_{k+1,...}$$

$$=$$

$$(1-(u_2+u_3+...)+(u_2u_3+u_2u_5+...)-(u_2u_3u_5+u_2u_3u_7+...)\pm...\pm u_2u_3...u_k)\,S_0$$

$$= (1+U_{p_k})(1+Q_{2,3,4,...})$$

$$= 1+U_{p_k}+Q_{2,3,4,...}+\;U_{p_k}Q_{2,3,4,...} = 1+Q_{k+1,...}$$

The one's cancel, and re-arranging we get

$$U_{p_k}Q_{2,3,4,...} = -\left(U_{p_k}+Q_{2,3,4,...,k}\right)$$

wherein we see that the application of U_{p_k} to the total initial system state $Q_{2,3,4,...}$ will invert U_{p_k} and $Q_{2,3,4,...,k}$, which is exactly what should happen (since the q's by nature invert their object), *but* there are no q's (only u's) in U_{p_k}, so this formulation of the relationships shows that the inversions [can validly be seen to] occur via the rotations

[13] Every m-vector, $m>1$, calculates (scalar) exclusive-or = *same/different,* via (-1)(+1) = (+1)(-1) = -1 and (+1)(+1) = (-1)(-1) = +1. Also encoded is A excludes B *vs*. A co-occurs with B.

of the Q_i by the U_i rather than by multiplication by -1 as in the $\overline{q}_i$ version. That is, both the wave and the particle views are simultaneously valid. Also, depending on the specific situation, the transition can be either reversible or irreversible.

Physically, U_{P_k} is the so-called quantum potential Ψ. As noted earlier, the $m > 1$ elements of this space can be constructed using the co-boundary operator, so (eg.) $\delta(u+v) = uv$ and $\delta(u+vw) = uvw$. Physically, this construction takes place dynamically and continually, and defines via the reversibility of its components the resonant structure of Ψ. But the interpretation of this structure as "the *quantum* potential" is restrictive, since, as we have seen, *any* set of unitary entities can be the elements of such a structure: call it instead the ***causal*** *potential*. The *causal* ***potential***.

The Object Concept

The individual U_i are the computational *objects* in this model, that is, entities having a persistent - though not necessarily permanent - existence. This persistence is signified by the fact of their unitarity: $U_i^2 = \pm 1$, meaning that U_i encompasses its own inverse, and thus can change without changing, so to speak. It does this by possessing a persistent *internal state* that is different from its *name*: ab is the name, while $+ab$ and $-ab$ both encode *and* display the internal state. [We discuss *naming* later.]

As a software structure, a given U_i has two modes of operation, one reversible and (hence) *space-like*, and the other irreversible and (hence) *time-like*. The latter captures the objects definable by contemporary programming languages (eg. Java - they're all the same on this), since a given Java object is only active when it is entered via one of its functional ports, and once that function has been carried out, the object is once again utterly passive. This scenario corresponds to U_i being operated upon by the idempotent $-1+U_i$. It is strings of such activations, from U_i to U_j to U_k, that constitute the processes defined by *Wait* and *Signal* (and by procedure calls generally).

The *space-like* aspect of a given U_i derives from its place in a surround of other similar objects, as defined by U_{p_k}. The changes in the surround sensed by the primitive 1-vectors x_i, x_j are combined pair-wise (via δ) to produce the 2-vector object $x_i x_j$, which in turn can be combined with x_k to produce the 3-vector object $x_i x_j x_k$, etc. Thus every m-vector, $1 < m < n$, is both the collector of impulses from "below" (ie. from its constituent boundary entities) and the distributor of its own (consequent) state to the higher-level objects of which it is itself a boundary entity. This upward ascent δ is close kin to the calculus operation of integration $\int$, and similarly, the reversed, downward flow of the manifestation of the hierarchy's potential corresponds to differentiation [both denoted by ∂].

Hierarchical (ie. structural) relationships and hierarchy traversal are space-like, since $\partial\delta = 1 = \delta\partial$ if the Cauchy-Riemann conditions $XU = YV$ and $YU = -XV$ hold, which they do if X, Y, U, V are reversible, which they are in the present case.[14]

[14]It can also be shown that $\partial_X^p Q = 0 = \delta_Q^q X$ for some $p, q > 1$, ie. "the boundary of the boundary equals zero", as does the co-boundary of the co-boundary. One needs the definitions $\partial(1) = 0 = \delta(1)$, since we're not taking ∂ as an operator over the algebra, but rather as an element of the algebra, nor using the (usual, grade-reducing) dot product for ∂. We invoke de Rham's theorem by analogy, not by right.

The upshot is that as software objects, the U_i are *always active*, at least conceptually. This trait, when combined with the reversibility conferred by unitarity, means that the U_i , seen from without, *oscillate*, and hence the entire hierarchy can be seen as a complex wave-like structure. The upward flow (and steadily decreasing frequency) performs a Fourier-like decomposition of the input vector, the primitive boundary $x_1 + x_2 + \ldots$, and the downward flow is a complex (but literal) reflection of that input. Notice that from an external point of view, the discrete particle-like aspect of the individual U_i has entirely disappeared from view ... one sees only waves and constructive and destructive interference - the *wave function* of the system as a whole![15]

From a programming point of view, every U_i is an instance of the *same* abstract object, namely one that δ-combines two other objects into one, and itself is δ-combinable into another such object. The general conceptual view is "abiding" rather than the usual "looking at". The communication regime is *broadcast-listen* rather than the ubiquitous and inherently time-like *request-reply* of virtually all programming languages and networked interactions.

Finally, the possession of an explicit wave-function U_P, a system's space-like aspect, constitutes a definitive criterion, otherwise lacking, of what it takes for a system to be "distributed". The dependence of the state of *every* locale on the state of *every other* locale, is both the ideal of a "distributed system" and *exactly* what a wave-based structure provides. Think of the waves in a bathtub - the height of every point on the surface is dependent on the heights of all the other points, and understand that this is a mere scalar version of the conceptual complexity that is organized in this fashion. Remote procedure call, ie. request-reply, is the *wrong* primitive for building truly distributed systems![16]

Discrete Physics

The recent book *The Origin of Discrete Particles* by Bastin and Kilmister [10] calculates the value of the inverse fine structure constant α^{-1} to one part in 10^7 on a closely reasoned and purely combinatorial basis.[17] The combinatorics of the present graded algebraic hierarchy match those of their Combinatorial Hierarchy (which is over $\mathbb{Z}_2 = \{0, 1\}$), which forms the structural basis for their calculation. Kilmister and the present author agree that there is most probably an isomorphism, though its exact form has not been investigated. Both structures are based on exclusive-or.

Bastin and Kilmister place the *process of constructing knowledge* at the center of their analysis:

The idea which underlies combinatorial physics is that of process. The most fundamental knowledge that we can have is of step-by-step unfolding of things; so in a sequence.

[15] [Three years later: Theorem (Parseval, 1799). The projection of a function F onto an orthogonal inner-product space (eg. $\mathscr{G}_n : \mathscr{O}(2^n)$) constitutes the Fourier decomposition of F.]

[16] The *right* primitive is $\mathsf{Co}[\mathsf{A}, \mathsf{B}]$, which blocks until states A and B co-occur. *JavaSpaces* is the right kind of platform for this. $\mathsf{Co}[\mathsf{A}, \mathsf{B}]$, and it's complement, $\mathsf{NotCo}[\mathsf{A}, \mathsf{B}]$, are the author's extensions to the *Linda* distributed programming paradigm, from which *JavaSpaces* derives. See [4]/Linda, and [6].

[17] Yielding 137.036011393... versus the latest empirical value, 137.035999710(96)... .

This is the kind of knowledge we have of quantum processes, and that fact becomes specially evident in the experimental techniques of high-energy physics. Such a process is necessarily combinatorial but not conversely. [p. 3]

The fundamental act in Bastin and Kilmister's analysis, the *empirical* act, is that of finding an entity in the otherwise entirely unknown surround and determining whether or not it is novel. They prefigure the derivation of $\mu = P_k S_0$, which similarly specifies "take an element of the otherwise unknown universe [ie. pick a u_k from S_0 and form $\bar{q}_k = -1 - u_k$] and compare it to what is already known [ie. form $\bar{q}_k P_{k-1} = (-1 - u_k)(1 + \sum_{m=1}^{k-1} \begin{pmatrix} U \\ m \end{pmatrix})$ and note if any $u_k U_j = \pm 1$], and if it is novel [ie. for no element of $\begin{pmatrix} U \\ m \end{pmatrix}$ is $u_k U_j = \pm 1$] , adjoin u_k to what is already known [ie. add u_k to $\begin{pmatrix} U \\ m \end{pmatrix}$, the P hierarchy in its sum-form]".

Note that the acquisition of this *explicit* knowledge is shown, via q's idempotence, to be an irreversible process.

Naming

There are still many loose ends in the frequency calculation, eg. when the frequencies f_i differ. Nevertheless it seems apparent that the appearance of the prime integers in physics is connected to the individual measurement process - that is, particular prime numbers are associated with particular individual processes. The product P defines these associations.

One can though make an *independent* argument for why the x_i, the 1-vector generators of the algebra, should be assigned, *literally*, prime-number frequency values. This follows from computer science, in particular from algorithms for achieving mutual exclusion between computational processes. As noted above, the computational primitives for accomplishing this are called, generically, ***Wait***(e) and ***Signal***(e) where e is some event/state. There are many ways to do this and algorithms for realizing *Wait* and *Signal* have a long history of interest and research.[18]

Relevant here is a mutual exclusion algorithm for distributed contexts, Lamport's *Bakery algorithm*, inspired by the take-a-number systems often found where there is a queueing problem. The nice technical feature of this algorithm is that it separates (the mechanism of) the determination of *access sequence* from (the mechanism of) *granting access* per se, and is thus well-suited to today's logically and physically distributed systems. The feature of interest in the present context is that the algorithm demonstrates the intimate relationship between the integers and the fundamental concept of the discreteness of events and their ordering into processes.

The key point is that the integers *also provide unique names* for the participating entities, and the issue of naming is, like mutual exclusion, a central concern in computer science, both practically (a name is a *de facto* address) and theoretically (granting a name opens the door for the *object concept* and the type and attributes of said objects).

[18] See the writings of C.A. Petri, E.W. Dijkstra, L.A. Lamport, and many others; and/or any good textbook on operating systems.

Thus the Bakery algorithm inspires the thought that the *same* numbering strategy could be used simultaneously for both sequencing and naming in an elegant marriage of functionalities.

Returning now to ζ and μ, we indeed see how an *integer*-based numbering in the sum-form of ζ is turned into a *prime*-based [and nearly Gödel] numbering of the hierarchical *m*-vectors in the sum-form U_P. Indeed, we see that the [integer-based] ordering ambiguities mentioned earlier are automatically swept aside to produce *unique, self-identifying m*-vector names that are namely *independent* of the ordering of the changes they connect *via* their co-boundary relationship (eg. in *ab*, flipping *a* and flipping *b*). It is particularly telling, in this context, that *every* possible combination over the u_i is automatically constructed, and uniquely named en route.

Finally, returning to the physics, having shown that the *mechanism* for accomplishing the phenomena uses (indeed, *needs*) the primes to construct the *literal names* of the unitary entities constituting the causal potential's structure, it is but a short and natural step to hypothesize that Nature herself uses this naming scheme "to keep track of things", and that, therefore, the 1-vector unitaries x_i physically *too* are identified by having wavelengths that are *prime* multiples of the Planck length $\tilde{h}$. By extension, the *m*-vector names $x_i \ldots x_j \ldots x_k$, where i, j, k here are primes, would indicate the *m*-vector's frequency components, and similarly but more intricately for compound unitaries like $a+b+ab$ and $ab+ac$.

And then, quite suddenly, the Fever broke.

Those wishing to avoid further infection should contemplate the definitions and truths regarding ζ as expressed by the Weil Conjectures in the *Appendix*, expressed with the admirable and exquisitely inscrutable clarity and precision that we all expect, and usually get, from our mathematical colleagues.

If you do not understand this appendix, you will likely be spared further trauma. On the other hand, a prophylactic application of these conjectures *can* limit the scope of further attacks, as indicated by the present case. And certainly, an ignorance of whatever-the-f Etale cohomology is will go a long ways too.[19]

If however you succeed in decoding the conjectures, there is a very good chance that you will instead be drawn into the very maw of the Riemann Hypothesis, namely over $\mathbb{R}$, the worst and most incurable form of Riemann Fever. Only a proof of the Hypothesis itself can cure it ... and forget the fame and the $\$10^6$ prizes - they'll be too late.

Few mathematical problems can lay claim to such a powerful combination of elementary nature, breadth of applications, and depth of theory inspired in the search for a proof.

Brian Osserman, *A Concise Account of the Weil Conjectures and Etale Cohomology [3]*

[19]Reader exercise: Exactly why, according to the conjectures, is the proof presented not news?

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References

1. Cipra, Barry. "Prime Formula Weds Number Theory and Quantum Physics". Science v. 274, 20 December 1996, p. 2014-5.

2. Derbyshire, John. *Prime Obsession - Bernhard Riemann and the Greatest Unsolved Problem in Mathematics*. Pub. Joseph Henry Press (National Academy of Sciences) 2003. ISBN 0-309-08549-7.

3. Osserman, Brian. "The Weil Conjectures".www.math.ucdavis.edu /~osserman/classes/256B/notes/sem-weil.pdf

4. Wikipedia. The Weil Conjectures.

5. Manthey, Michael. "Synchronization - the Font of Physical Structure". International Journal of Anticipatory Systems, Daniel Dubois, Ed. Issn 1373 - 5411, v. 20-21-22, IJCAS/BCS Symposium, Liege, Belgium. August 6-11, 2007.

6. Manthey, Michael. "A Combinatorial Bit-Bang Leading to Quaternions". Proc. 1997 Helsinki Conference on Emergence, Hierarchy, and Organization (ECHO III). arXiv.org > quant-ph/9809033v1.

7. Hestenes, David. *New Foundations for Classical Physics*, 2nd edition. Kluwer, 1999. ISBN 0-7923-5514-8.

8. Lounesto, Pertti. *Clifford Algebras and Spinors*. London Mathematical Society Lecture Note Series #286, 2nd edition. Cambridge University Press, 2001. ISBN 0-521-00551-5.

9. Hestenes, David. *Clifford Algebra to Geometric Calculus - A Unified Language for Mathematics and Physics*. Reidel/Kluwer, 1987. ISBN 90-277-2561-6.

10. Bastin, Ted and Kilmister, Clive. *The Origin of Discrete Particles*. World Scientific, Series on Knots and Everything, L. Kauffman, Ed. ; v. 42, 2009. ISBN-13: 978-981-4261-67-8.

11. Dantzig, Tobias and Mazur, Joseph. *Number - the Language of Science*. 1930(2007). ISBN-10: 0452288118; ISBN-13: 978-0452288119.

This paper is dedicated, with profound gratitude, to Clive Kilmister (1924-2010).

See www.tauquernions.org for other papers by the author.

Appendix - *The Weil Conjectures [4]*

Suppose that X is a non-singular n-dimensional projective algebraic variety over the field F_q with q elements. The zeta function $\zeta(X,s)$ of X is by definition

$$\zeta(X,s) = \exp\left(\sum_{m=1}^{\infty} \frac{N_m}{m}(q^{-s})^m\right)$$

where N_m is the number of points of X defined over the degree m extension $F_q m$ of F_q.

The conjectures state:

1. *Rationality*. $\zeta(X,s)$ is a rational function of T = q^{-s}. More precisely, $\zeta(X,s)$ can be written as a finite alternating product

$$\prod_{i=0}^{2n} P_i(q^{-s})^{(-1)^{i+1}} = \frac{P_1(T)\cdots P_{2n-1}(T)}{P_0(T)\cdots P_{2n}(T)}$$

where each $P_i(T)$ is an integral polynomial. Furthermore,

$$P_0(T) = 1 - T,\ P_{2n}(T) = 1 - q^n T \quad ;$$

and for $1 \leq i \leq 2n-1$, $P_i(T)$ factors over C as $\prod_j (1 - \alpha_{ij} T)$ for some numbers $\alpha_{ij}\ldots$.

2. *Functional equation and Poincaré duality*.

$$\zeta(X, n-s) = \pm q^{\frac{nE}{2} - Es} \zeta(X,s)$$

or equivalently

$$\zeta(X, \tfrac{1}{q^n T}) = \pm q^{\frac{nE}{2}} T^E \zeta(X,T)$$

where E is the Euler characteristic of X. In particular, for each i, the numbers $\alpha_{2n-i,1}$, $\alpha_{2n-i,2},\ldots$ equal the numbers $\frac{q^n}{\alpha_{i,1}}, \frac{q^n}{\alpha_{i,2}}, \ldots$ in some order.

3. *Riemann hypothesis*. $|\alpha_{i,j}| = q^{\frac{i}{2}}$ for all $1 \leq i \leq 2n-1$ and all j. This implies that all zeros of $P_k(T)$ lie on the "critical line" of complex numbers s with real part $\frac{k}{2}$.

4. *Comparison*. If X is a (good) "reduction mod p" of a non-singular projective variety Y defined over a number field embedded in the field of complex numbers, then the degree of P_i is the ith Betti number of the space of complex points of Y.

The Weil Conjectures were proven to be true by Pierre Deligne in 1973.